Les structures exactes

par

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Durant mon doctorat, j'ai travaillé principalement dans le cadre des catégories exactes. C'est donc le sujet central de quatre de mes articles ([BHLR18], [HR19], [BHT20] et [BBGH20]). De plus, j'ai travaillé sur les généralisations des catégories exactes, comme les catégories partiellement exactes, les catégories extriangulées et les catégories partiellement extriangulées. J'ai aussi publié deux autres articles, l'un sur les paires de cotorsion d'une catégorie extriangulée [HS20] et un autre sur les catégories additives et les structures exactes dans le cadre de l'analyse fonctionnelle [HSW20]. Dans cette thèse, j'ai choisi trois de mes articles étudiant les structures exactes et partiellement exactes. J'ai étudié plusieurs sujets relatifs à la structure exacte d'une catégorie exacte. De plus, j'ai étudié comment certaines propriétés changent en réduisant ou en élargissant la structure exacte considérée.

On considère l'ensemble partiellement ordonné $(Ex(\mathcal{A}), \subseteq)$ formé par toutes les structures exactes données sur une certaine catégorie additive fixe et on prouve qu'elles forment un treillis borné et complet. En parallèle, on étudie le treillis des sous-bifoncteurs additifs fermés du foncteur Ext^1 et on construit un isomorphisme de treillis entre ces deux derniers. On étudie aussi le treillis de sous-bifoncteurs additifs en général et on introduit les structures partiellement exactes et partiellement extriangulées, on étudie leurs treillis et on résume les liens entre tous ces treillis.

On considère les sous-objets admissibles relativement à une structure exacte et on propose

des nouvelles notions générales d'intersection et de sommes définies pour toute catégorie exacte. En utilisant ces notions on considère les catégories d'Artin-Wedderburn exactes et on généralise le théorème de Jordan-Hölder. Dans le cas d'une catégorie de modules sur une algèbre de Nakayama, les catégories d'Artin-Wedderburn exactes sont exactement les catégories exactes de Jordan-Hölder. Sur ces catégories, on définit la longueur de Jordan-Hölder. Puis on la généralise en définissant la fonction de longueur pour toute catégorie exacte et, comme application, on généralise la mesure de Gabriel-Roiter pour n'importe quelle catégorie exacte.
L'étude des sous-objets et des morphismes admissibles d'une catégorie exacte mène aussi à de nouvelles caractérisations des catégories quasi-abéliennes et abéliennes.

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Finalement, je remercie toutes les personnes qui m'ont soutenue le long de mon doctorat.

Souheila Hassoun

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INTRODUCTION

En 1934, R. Baer introduit le foncteur d'extension Ext pour les groupes abéliens et en 1940 il définit une opération de somme sur les suites exactes courtes, qui utilise le produit fibré et la somme amalgamée, et permet d'avoir une description explicite de la structure de groupe abélien sur l'ensemble des classes d'équivalence des extensions. En 1954, N. Yoneda prouve le théorème de classification donnant une correspondance entre les classes d'équivalence des $n-$extensions de B par A et les éléments du groupe abélien $Ext^n(A, B)$. Alors, dans [Y54], Yoneda est le premier à donner une indication claire sur comment définir les foncteurs d'extensions d'une manière abstraite. En 1955, D. Buchsbaum prouve l'existence du foncteur Ext sur une catégorie exacte admettant assez d'objets projectifs et injectifs. Plus tard en 1957, inspiré par l'idée de Yoneda, Buchsbaum définit le foncteur d'extension sur une catégorie exacte, d'une manière abstraite, sans utiliser les résolutions projectives ni injectives. On cite aussi que É-J. Cartan et S. Eilenberg discutent dans leur livre [CE56], apparu en 1956, l'algèbre homologique et généralisent les notions de produit tensoriel et de groupes d'extensions pour les catégories de modules.

En 1958, G. Hochschild introduit l'idée d'homologie relative en discutant l'analogue du foncteur Ext mais relatif aux modules sur un sous-anneau de l'anneau basique des opérateurs. Dans la même période de temps, d'autres travaux par Heller et Harrison discutent des problèmes similaires et il devient de plus en plus naturel de considérer le foncteur Ext, associé à la structure exacte d'une catégorie exacte spécifique.

L'idée de l'homologie relative est de choisir une classe d'extensions dans une catégorie additive. On note qu'après une série d'articles par J-P.Schneiders [Sch99], D. Sieg et S.-A. Wegner [SW11], S. Crivei [Cr11, Cr19] sur la structure exacte maximale sur une catégorie additive donnée, W. Rump réussit, en 2011 [Ru11, Ru15], à prouver généralement l'existence d'une unique structure exacte maximale sur n'importe quelle catégorie additive. Depuis ce résultat, on peut voir l'idée d'homologie relative comme une réduction de la structure exacte maximale. Partant de cette structure maximale, on choisit une sous-structure exacte, c'est-à-dire une sous-classe de suites exactes courtes, ou deux familles de monomorphismes et d'épimorphismes, qui satisfait aux axiomes nécessaires. Puis on étudie la théorie d'homologie relative à cette structure fixée. Notamment, en 1961, C.R. Butler et G. Horrocks étudient dans [BuHo61] l'homologie relative sur les catégories abéliennes. Ils étudient comment les propriétés des foncteurs dérivés changent sous la réduction des structures exactes.

En 1972, D. Quillen introduit sa formulation de la $K-$théorie supérieure qui est considérée comme sa contribution la plus célèbre. La $K-$théorie supérieure a montré son efficacité dans la formulation et la résolution de problèmes majeurs en algèbre, en particulier en théorie des anneaux et en théorie des modules. En 1978, il obtient la médaille Fields pour ses travaux.

Dans ce travail, on utilise la définition axiomatique d'une structure exacte introduite par D. Quillen en 1973 dans [Qu73]. Une catégorie exacte est une paire $(\mathcal{A}, \mathcal{E})$ constituée d'une catégorie additive $\mathcal{A}$ et d'une structure exacte $\mathcal{E}$.

Une catégorie préadditive $\mathcal{A}$ est une catégorie dans laquelle chaque ensemble de morphismes $Hom_{\mathcal{A}}(A, B)$ entre deux objets A et B est un groupe abélien et pour laquelle la composition des morphismes est biadditive. Une catégorie préadditive est dite additive s'il existe un objet zéro et si chaque couple d'objets admet un biproduit.

On rappelle la définition d'une structure exacte de Quillen depuis le livre [GR92, Bü] écrit par P. Gabriel et A.W. Roiter :

Définition 0.1. Soit $\mathcal{A}$ une catégorie additive. Une suite exacte courte (i,d) dans $\mathcal{A}$ est une paire de noyau-conoyau, c'est à dire une paire de morphismes composables, telle que i est le noyau de d et d est le conoyau de i. Si on choisit une classe $\mathcal{E}$ de paires noyau-conoyau ou de suites exactes courtes dans $\mathcal{A}$, un monomorphisme admissible est un morphisme i pour lequel il existe un morphisme d tel que $(i,d) \in \mathcal{E}$. Un épimorphisme admissible est définit dualement. Une structure exacte est une classe de suites exactes courtes stable sous les isomorphismes et qui satisfait les six axiomes suivants :

(E0) Pour tout objet A de $\mathcal{A}$, le morphisme identité 1_A est un monomorphisme admissible

(E0)^{op} Pour tout objet A de $\mathcal{A}$, le morphisme identité 1_A est un épimorphisme admissible

(E1) La classe de tous les monomorphismes admissibles est fermée sous la composition

(E1)^{op} La classe de tous les épimorphismes admissibles est fermée sous la composition

(E2) La somme amalgamée d'un monomorphisme admissible i avec un morphisme arbitraire t existe et produit un monomorphisme admissible s_C :

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "t"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, "s_B"] \\ C \arrow[r, tail, "s_C"'] & S \end{tikzcd}$$

(E2)^{op} Le produit fibré d'un épimorphisme admissible h avec un morphisme arbitraire t existe et produit un épimorphisme admissible p_B

$$\begin{tikzcd} P \arrow[r, two heads, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\text{PB}"] & B \arrow[d, "t"] \\ A \arrow[r, two heads, "h"'] & C \end{tikzcd}$$

Cette définition d'une structure exacte, basée sur six axiomes, est raffinée plus tard par B. Keller la réduisant à quatre axiomes à la place dans [Ke91].

Plus récemment, M. Auslander et Ø.Solberg publient une série de travaux sur l'homologie relative. Dans la première [AS93], en 1993, ils discutent comment appliquer l'homologie relative à la théorie des représentations. Alors que dans la deuxième partie [AS05], apparue en 2005, ils développent la théorie de co-inclinaison relative pour les catégories de modules sur une algèbre d'Artin.

Notre approche pour l'étude des catégories exactes utilise cette définition axiomatique de Quillen. Pour généraliser des résultats des catégories de modules et des catégories abéliennes en général, on choisit d'écrire nos preuves en utilisant directement les axiomes explicites d'une catégorie exacte plutôt que d'utiliser les théorèmes de plongement. On rappelle qu'il existe plusieurs théorèmes de plongement. Le Théorème de plongement de Freyd-Mitchell ([HaRo64, Mi64]) permet d'étendre les résultats des catégories de modules vers les catégories abéliennes. Le théorème de plongement de Gabriel-Quillen dit qu'à l'aide des foncteurs de yoneda pleins et fidèles, on peut plonger toute petite catégorie exacte $(\mathcal{A}, \mathcal{E})$ dans la catégorie abélienne $\mathcal{B}$ formée par les foncteurs exacts à gauche de $\mathcal{A}^{op} \to Ab$ tel que l'image essentielle est stable pour les extensions. Dans ce cas une suite courte est exacte dans $\mathcal{A}$ si et seulement si elle est exacte dans $\mathcal{B}$. Ce théorème fournit des bonnes intuitions et sert à réduire l'algèbre homologique dans des catégories exactes à des catégories abéliennes. Dans notre travail, on décide d'utiliser une approche différente pour prouver des résultats sur des catégories exactes sans utiliser aucun plongement. Cette nouvelle approche est déjà discutée et défendue par T.Bühler dans le résumé suivant [Bü]. Dans son résumé, il parle aussi de l'importance des catégories exactes et de leur utilité. Il cite aussi plusieurs applications intéressantes des catégories exactes, notamment dans les domaines de géométrie algébrique et de l'analyse algébrique.

Le type d'une catégorie additive (résumé par le diagramme de la page 7) joue un rôle très important dans notre étude des catégories exactes. Souvent, la différence entre ces types revient aux propriétés des suites exactes courtes dans la catégorie donnée. L'existence d'une structure exacte ou autrement dit le fait qu'une certaine classe de suites exactes courtes forme une structure exacte sur une catégorie, dépend du fait que ces suites satisfont ou pas aux axiomes d'une structure exacte. C'est ici que le type de la catégorie additive vient jouer son rôle. Les propriétés de la catégorie vont permettre de déterminer les choix de structures exactes qu'on a dans cette catégorie. On rappelle alors les définitions des sous classes les plus connues et utilisées des catégories additives. On commence par les catégories partiellement idempotentes complètes. Ce type de catégories additives est assez général vu que toute catégorie additive admet une complétion karoubienne de ce type (voir [Ka68, page 75]) :

Définition 0.2. Une catégorie additive est dite partiellement idempotente complète si toute section admet un conoyau ou toute rétraction admet un noyau.

Définition 0.3. Une catégorie additive est dite idempotente complète si tout morphisme idempotent e (c'est-à-dire un morphisme tel que $e^2 = e$) admet un noyau ou un conoyau.

Définition 0.4. Une catégorie additive est dite semi-abélienne si elle est pré-abélienne (c'est-à-dire qu'elle admet tous les noyaux et les conoyaux) et que le morphisme canonique

$$\bar{f} : Coimf \to Imf$$

associé à tout morphisme $f : A \to B$, est un monomorphisme et un épimorphisme. Ici $\bar{f}$ est l'unique morphisme rendant commutatif le diagramme suivant

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \\ Coimf \arrow[r, dotted, "\bar{f}"'] & Imf \arrow[u, dash, "i"'] \end{tikzcd}$$

c'est à dire tel que $f = i \circ \bar{f} \circ p$, $i : Imf \to B$ est l'inclusion canonique et $p : A \to Coimf$ est la projection canonique.

Définition 0.5. [RW77, p.524]Un noyau (A, f) d'un certain morphisme, si il existe, est dit *semi-stable* si pour toute somme amalgamée

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "s_B"] \\ C \arrow[r, "s_C"'] & S \end{tikzcd}$$

le morphisme s_C est aussi un noyau (d'un autre morphisme). On définit dualement un conoyau semi-stable. Une suite exacte courte

$$A \xrightarrow{i} B \xrightarrow{d} C$$

est dite *stable* si i est un noyau semi-stable et d est un conoyau semi-stable. On note $\mathcal{E}_{sta}$ la classe de toutes les suites exactes courtes *stables*.

Définition 0.6. Soit $\mathcal{A}$ une catégorie additive pré-abélienne. Un morphisme f est dit strict si son morphisme canonique $\bar{f}$ est un isomorphisme. Une suite exacte courte $A \xrightarrow{i} B \xrightarrow{d} C$ est dite stricte si i est strict ou d est strict. On note par $\mathcal{E}_{str}$ la classe de toutes suites exactes courtes strictes dans $\mathcal{A}$.

Maintenant on rappelle la définition des catégories *quasi-abéliennes*. Ces catégories apparaissent beaucoup en analyse fonctionnelle où elles jouent un rôle très important. Parmi les exemples de telles catégories, on cite la catégorie des espaces de Banach, la catégorie des espaces de Fréchet, la catégorie des espaces localement convexes et la catégorie des espaces semi-normés.

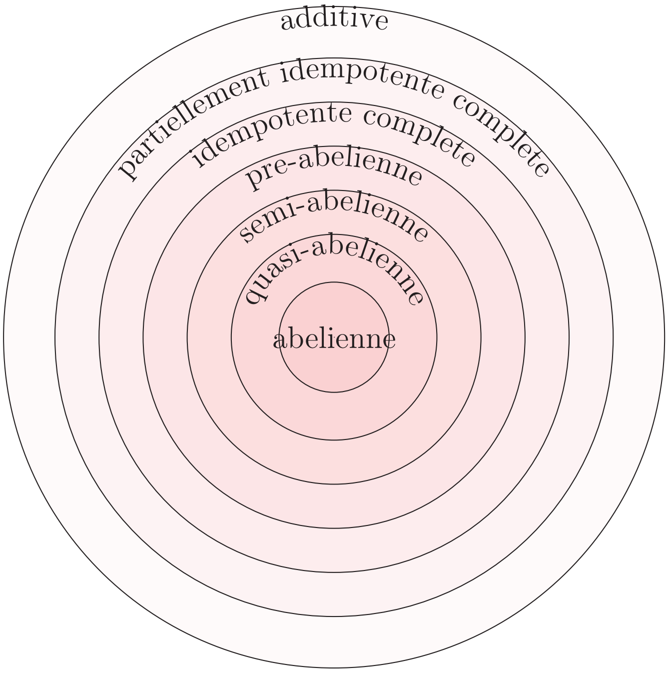
Définition 0.7. Une catégorie additive $\mathcal{A}$ est *quasi-abélienne* si elle est *pré-abélienne* et que tous les noyaux et les conoyaux sont *semi-stables*.

Remarque 0.8. Une catégorie additive $\mathcal{A}$ est quasi-abélienne quand elle est pré-abélienne et que le produit fibré d'un épimorphisme strict (voir 0.6 ou [BHLR18, Définition 2.7]) avec un morphisme arbitraire est aussi un épimorphisme strict, et que la somme amalgamée d'un monomorphisme strict est un monomorphisme strict.

Finalement, on rappelle la définition des catégories abéliennes, une sous-classe populaire et très importante des catégories additives. Parmi les exemples de telles catégories, on cite les catégories des modules sur une algèbre et la catégorie des groupes abéliens.

Définition 0.9. Une catégorie additive $\mathcal{A}$ est dite *abélienne* si elle est *pré-abélienne* et que tous ses morphismes sont stricts.

Voici un diagramme qui résume les liens entre toutes les sous-classes des catégories additives qu'on vient de définir :



Pour toute catégorie additive, il existe au moins deux structures exactes : la structure exacte minimale qu'on note $\mathcal{E}_{min}$ et la structure exacte maximale qu'on note $\mathcal{E}_{max}$. Il arrive que ces deux structures coïncident, c'est le cas de toute catégorie triangulée (une

catégorie additive qui admet une triangularisation, c'est à dire une classe de triangles satisfaisant certains axiomes, voir par exemple [Ne90]). Alors, pour être plus précis, il existe au moins une structure exacte pour toute catégorie additive .
La structure exacte minimale, inclue dans toute autre structure exacte, est toujours donnée par la classe de toutes les suites exactes scindées, c'est-à-dire toutes les suites exactes courtes isomorphes à une suite de la forme suivante :

$$A \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} A \oplus B \xrightarrow{[01]} B.$$

L'existence d'une unique structure exacte maximale pour toute catégorie additive est établie par W. Rump en 2011 dans [Ru11], et vient après une série de travaux sur le sujet menée par plusieurs mathématiciens. Alors que la structure exacte minimale est toujours donnée par les suites exactes courtes scindées, la structure maximale dépend du type de la catégorie additive. Pour toute catégorie quasi-abélienne, on sait que la structure maximale est donnée par toutes les suites exactes courtes (voir [Sch99]). D. Sieg et S-A. Wegner montrent en 2011 dans [SW11] que la classe de toutes les suites exactes courtes stables $\mathcal{E}_{sta}$ forment la structure exacte maximale pour toute catégorie pré-abélienne. Ce résultat est généralisé par S. Crivei en 2012 dans [Cr11] pour l'étendre à toute catégorie partiellement idempotente complète. Puis il a caractérisé, dans [Cr19], les catégories additives pour lesquelles la structure stable forme la structure maximale. Notant que $\mathcal{E}_{sta}$ ne forme pas la structure maximale pour n'importe quelle catégorie additive, un contre-exemple est donné par W. Rump dans [Ru15]. Dans [Ru11], la preuve de W. Rump sur l'existence d'une unique structure exacte maximale utilise les structures exactes à gauche et à droite, et assure l'existence abstraite. Jusqu'à maintenant, on ne peut pas décrire explicitement $\mathcal{E}_{max}$ pour des catégories additives générales.
Entre la structure exacte minimale et la structure exacte maximale, il peut exister plusieurs autres structures exactes. Voici deux exemples intéressants :

Exemple 0.10. [Qu73][Bri07, section 4]Une catégorie additive $\mathcal{A}$ munie de la classe des suites exactes courtes strictes $\mathcal{E}_{str}$ forme une catégorie exacte $(\mathcal{A}, \mathcal{E}_{str})$.

Exemple 0.11. [Ru08]Soit $A = kQ/I$ le quotient de l'algèbre des chemins KQ sur un corps k donnée par le carquois Q suivant et l'idéal I engendrée par les relations de commutativité :

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] & 3 \arrow[l] \arrow[d] \\ 4 \arrow[r] & 5 & 6 \arrow[l] \end{tikzcd}$$

Dans ce cas, l'algèbre est inclinée de type E_6. On considère la catégorie des $A-$modules projectifs de type fini $\mathcal{A} = A-proj$. Cette catégorie $\mathcal{A}$ constitue le premier exemple d'une catégorie semi-abélienne qui n'est pas quasi-abélienne. Alors la classe des suites exactes courtes stables forme la structure exacte maximale : $\mathcal{E}_{max} = \mathcal{E}_{sta}$.

La possibilité de choisir différentes structures exactes non seulement enrichit l'étude des catégories exactes, mais est aussi nécessaire dans plusieurs contextes. Cela nous offre plus de flexibilité, même au sein d'une catégorie abélienne. Souvent, quand on parle de catégorie abélienne, on parle de la catégorie exacte formée par une catégorie additive abélienne et sa structure exacte maximale formée par toutes les suites exactes courtes de la catégorie $\mathcal{E}_{all}$. Il est important, même dans le cadre des catégories abéliennes, de réduire cette structure et considérer toutes les structures exactes possibles entre $\mathcal{E}_{min}$ et $\mathcal{E}_{max} = \mathcal{E}_{all}$.

Pour une catégrie additive fixée, on considère toutes les structures exactes possibles sue celle-ci. L'ensemble de toutes ces structures exactes, noté $Ex(\mathcal{A})$, forme un ensemble partiellement ordonné par l'inclusion des classes. De plus, on démontre que cet ensemble forme un treillis complet et borné. Sous certaines conditions supplémentaires, c'est même un treillis booléen. C'est vrai pour les catégories de représentations d'un carquois fini de type Dynkin. C'est le cas par exemple du treillis $Ex(\mathcal{A})$ pour $\mathcal{A} = \operatorname{rep} Q$ de la catégorie des représentations du carquois suivant de type A_3 :

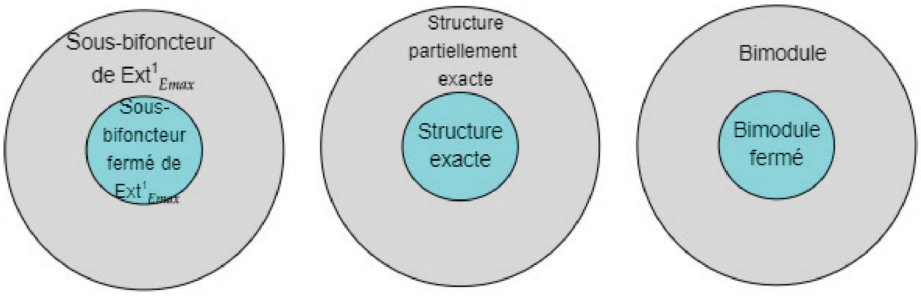
$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

Voici un diagramme qui représente $Ex(\operatorname{rep} Q)$ dans ce cas :

$$\begin{tikzcd}
 & \mathcal{E}_3 \arrow[rr] \arrow[dl] & & \mathcal{E}_{1,3,5} \arrow[dl] \\
\mathcal{E}_{2,3,4} \arrow[rr] & & \mathcal{E}_{max} & \\
 & \mathcal{E}_{min} \arrow[rr] \arrow[dl] \arrow[uu] & & \mathcal{E}_1 \arrow[dl] \arrow[uu] \\
\mathcal{E}_2 \arrow[rr] \arrow[uu] & & \mathcal{E}_{1,2} \arrow[uu] &
\end{tikzcd}$$

On établit aussi une structure de treillis sur l'ensemble des sous-bifoncteurs additifs *fermés* (au sens de Butler et Horrocks [BuHo61]) et on démontre que ces deux treillis sont isomorphes. On démontre aussi que ces deux treillis sont également isomorphes à deux autres treillis.

En oubliant l'un des axiomes dans la définition d'une structure exacte et en ajoutant la condition d'être stable pour les sommes directes, on introduit les structures partiellement exactes, une généralisation des structures exactes. Une structure partiellement exacte correspond à un sous-bifoncteur additif, pas nécessairement fermé, du foncteur d'extension Ext (voir [Y54]). On démontre que ces structures forment un treillis, et que ce treillis est isomorphe au treillis de tous les sous-bifoncteurs additifs de Ext. Dans le but est de simplifier l'étude de ces treillis on propose un troisième treillis. Celui-ci est le treillis de tous les bimodules sur l'algèbre d'auslander, qui est isomorphe aux deux autres treillis. Le diagramme suivant résume les ensembles partiellement ordonnés qu'on a étudiés :



On étudie les propriétés de ces nouvelles structures, et on montre que plusieurs des caractéristiques des structures exactes restent valides. Ainsi, on prouve une version de l'axiome obscure de Quillen [Qu73] mais pour les catégories partiellement exactes. De plus, on généralise l'existence d'une unique structure partiellement exacte maximale pour une catégorie additive partiellement idempotente complète et on généralise la caractérisation de [Cr19].
On introduit les notions de structure partiellement exacte à droite et partiellement exacte à gauche, de façon qu'elles généralisent les structures exactes à gauche et à droite introduites dans [BC12]. On prouve que toute structure partiellement exacte se construit à partir d'une paire formée par une structure partiellement exacte à droite et une autre à gauche, et vice versa.
On généralise encore ce concept en définissant les structures partiellement extriangulées. Ces dernières généralisent en même temps les structures extriangulées et les structures partiellement exactes. On note que les catégories extriangulées, introduites par Nakaoka et Palu en 2018 dans [NP19], généralisent simultanément les catégories exactes et triangulées.

Un autre sujet abordé durant mon doctorat est la notion de longueur sur une catégorie exacte. Pour une catégorie abélienne, et plus précisément pour la catégorie de modules de dimension finie sur une algèbre, la longueur est une application qui associe à chaque module un nombre entier positif qui généralise sa dimension en tant qu'espace vectoriel. Cette longueur est définie comme l'unique longueur d'une suite de composition pour cet objet, dont l'unicité rend la fonction de longueur correctement définie. C'est une conséquence du fameux théorème de Jordan-Hölder valide pour les catégories abéliennes. Ce théorème est dû aux mathématiciens C. Jordan et O. Hölder (1869-1889). Ce résultat peut être prouvé en utilisant le lemme de Schreier, et c'est pourquoi il est aussi parfois appelé le théorème de Jordan-Hölder-Schreier. En 2006, Baumslag donne une preuve courte et plus simple de ce théorème dans [Ba06], en utilisant les notions d'intersection de somme des sous-objets.

On considère la propriété de Jordan-Hölder relativement à une structure exacte donnée $\mathcal{E}$, et on étudie les catégories exactes $(\mathcal{A}, \mathcal{E})$ suivantes :

Définition 0.12. [BHT20, Définition 5.1] Soit $(\mathcal{A}, \mathcal{E})$ une catégorie exacte. Une $\mathcal{E}$–suite de composition d'un objet X de $\mathcal{A}$ est une suite

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{n-2}}{\rightarrowtail} X_{n-1} \overset{i_{n-1}}{\rightarrowtail} X_n = X \tag{1}$$

où les i_l sont des *monomorphismes admissibles propres* avec des conoyaux $\mathcal{E}$–simples (voir [BHLR18, Définition 3.3]). On dit qu'une catégorie exacte $(\mathcal{A}, \mathcal{E})$ admet *la propriété ($\mathcal{E}$–)Jordan-Hölder* ou est *une catégorie exacte de Jordan-Hölder* si n'importe quelles deux $\mathcal{E}$–suites de composition finies d'un objet X de $\mathcal{A}$

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{m-2}}{\rightarrowtail} X_{m-1} \overset{i_{m-1}}{\rightarrowtail} X_m = X$$

et

$$0 = X'_0 \overset{i'_0}{\rightarrowtail} X'_1 \overset{i'_1}{\rightarrowtail} \dots \overset{i'_{n-2}}{\rightarrowtail} X'_{n-1} \overset{i'_{n-1}}{\rightarrowtail} X'_n = X$$

sont équivalentes, c'est à dire que quand ces $\mathcal{E}$–suites de composition finies existent, elle admettent la même longueur et les mêmes facteurs de composition (c'est à dire les conoyaux $\mathcal{E}$–simples des monomorphismes admissibles considérés) à permutation près.

Comme plusieurs preuves de ce théorème dans le cas abélien utilisent les notions d'intersection et de somme, notre approche est de commencer par généraliser ces notions. L'approche naïve est d'adapter l'intersection et la somme abélienne en utilisant le produit fibré et la somme amalgamée. On considère des nouvelles catégories exactes, qu'on a introduites dans [HR19], les catégories exactes admettant des intersections admissibles, c'est-à-dire les catégories exactes satisfaisant à un axiome de plus, qu'on note l'axiome (AI). On découvre une nouvelle caractérisation des catégories quasi-abéliennes, en montrant qu'une catégorie exacte admet des intersections admissibles (AI) si et seulement si elle est quasi-abélienne. On considère aussi les catégories exactes admettant des intersections admissibles et des sommes admissibles qu'on note (AIS) d'après [HR19], et on démontre que celles-ci sont exactement les catégories abéliennes. De plus, on donne d'autres nouvelles caractérisations des catégories abéliennes en utilisant des propriétés des morphismes admissibles dans une catégorie additive.
Ces caractérisations montrent qu'adapter l'intersection et la somme en utilisant le produit fibré et la somme amalgamée ne se généralise pas pour toutes les catégories exactes en général. Suivant ces observations, on propose des nouvelles définitions d'intersection et de somme valides pour toute catégorie exacte. Généralisant les notions d'intersection et de somme usuelles qu'on connait pour les catégories abéliennes, ces nouvelles sont définies comme des ensembles de sous-objets admissibles et non plus par un seul sous-objet unique.
Ces nouvelles notions générales d'intersection et de sommes nous permettent de généraliser la notion de radical et ainsi de définir les catégories d'Artin-Wedderburn exactes. On prouve que toute catégorie exacte d'Artin-Wedderburn satisfait à la propriété de Jordan-Hölder.
Maintenant, on restreint notre étude au cas des catégories de modules sur une algèbre de Nakayama. Après avoir décrit toutes les structures exactes sur de telles catégories, on caractérise toutes les catégories exactes d'Artin-Wedderburn en démontrant qu'elles sont

exactement les catégories exactes de Jordan-Hölder.
Une catégorie exacte de Jordan-Hölder admet une fonction de longueur de Jordan-Hölder. Cette dernière possède de bonnes propriétés et améliore la fonction de longueur qu'on définit en général sur toute catégorie exacte par la longueur de la filtration maximale.
On étudie ces fonctions de longueur relatives dépendant du choix d'une structure exacte sur une catégorie additive fixe et les objets relativement artiniens et noethériens. Puis on montre que la longueur relative d'un certain objet se réduit en réduisant la structure exacte.
Comme application de la fonction de longueur, on généralise la mesure de Gabriel-Roiter, définie et formulée par P. Gabriel, dans [Gab73], après être utilisée par A.V. Roiter dans la preuve de la première conjecture de Brauer-Thrall [Ro68], dans le cadre des catégories abéliennes. On étudie la mesure de Gabriel-Roiter sur une catégorie exacte et on la compare aux travaux déjà publiés dans le cas abélien [Ri05], [Ri06], [Kr07], [Kr11]. Enfin, on observe que malgré les différences, plusieurs de ses propriétés restent satisfaites dans ce cadre plus général.

CHAPITRE 1

Réduction des structures exactes

Cet article est écrit en anglais dans sa version originale, et est accessible dans l'annexe A de cette thèse. Écrit par Th.Brüstle, D.Langford, S.Roy et moi même, cet article est publié dans le Journal suivant : *Journal of Pure and Applied Algebra.*

Dans cet article, on propose de fixer la catégorie additive en gardant tous les objets originaux, mais en choisissant une classe de morphismes admissibles. Partant de la structure exacte maximale, on peut la réduire en choisissant une sous-structure exacte $\mathcal{E} \subseteq \mathcal{E}_{max}$ jusqu'à ce qu'on se retrouve avec la structure exacte minimale $\mathcal{E}_{min} \subseteq \mathcal{E}$. On remarque que l'ensemble de toutes ces structures exactes est un ensemble partiellement ordonné par inclusion des classes et on décide de l'étudier. On le munit alors de la structure de treillis suivante et on montre que ce treillis est toujours borné et complet pour ces opérations :

Théorème 1.1. L'ensemble partiellement ordonné de toutes les structures exactes sur une catégorie additive $\mathcal{A}$ est un treillis complet et borné :

$$(Ex(\mathcal{A}), \subseteq, \bigwedge, \bigvee)$$

où *l'infimum* ou la borne inférieure (le plus grand des minoranst) $\bigwedge$ est donné par l'intersection des classes $\mathcal{E}_\omega \bigwedge \mathcal{E}_{\omega'} = \mathcal{E}_\omega \cap \mathcal{E}_{\omega'}$, et *le suprémum* ou la borne supérieure (le plus petit des majorants) $\bigvee$ est défini par

$$\mathcal{E}_\omega \bigvee \mathcal{E}_{\omega'} = \bigcap \{\mathcal{E} \in Ex(\mathcal{A}) \mid \mathcal{E}_\omega \subseteq \mathcal{E}, \mathcal{E}_{\omega'} \subseteq \mathcal{E}\}.$$

La borne supérieure proposée est correctement définie pour toute catégorie additive, grâce au résultat sur l'existence d'une structure exacte maximale prouvé par W. Rump en 2011 dans [Ru11]. Il suffit après de vérifier que ces opérations vérifient les conditions nécessaires pour former un $\bigwedge$-semi-treillis et un $\bigvee$-semi-treillis en même temps.

Notre deuxième but est d'étudier comment la réduction d'une structure exacte, c'est à dire le choix d'une plus petite structure exacte dans le treillis $Ex(\mathcal{A})$, affecte les propriétés des notions relatives. Plus précisément, on répond à la question suivante :

Question 1.2. Comment la fonction de longueur change en réduisant la structure exacte ?

Pour répondre à cette question, on commence par définir la fonction de longueur pour toute catégorie exacte essentiellement petite. Vu que la propriété de Jordan-Hölder n'est pas toujours satisfaite, on ne peut pas simplement considérer la longueur d'une suite de composition relative. Alors, on décide de choisir la longueur maximale d'une filtration de sous-objets admissibles.

Définition 1.3. Soit $(\mathcal{A}, \mathcal{E})$ une catégorie exacte. On définit la $\mathcal{E}-$fonction de longueur $l_\mathcal{E} : Obj\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ comme la borne supérieure des longueurs des chaines de monomorphismes admissibles propres (qui ne sont pas des isomorphismes). Alors, pour un objet X de $(\mathcal{A}, \mathcal{E})$, on pose $l_\mathcal{E}(X) = n \in \mathbb{N} \cup \{\infty\}$, qui est la longueur maximale d'une chaine de monomorphismes admissibles

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_{n-1} \rightarrowtail X_n = X.$$

Dans ce cas, on dit que X est de $\mathcal{E}-$longueur finie. Si une telle borne n'existe pas, on dit que X est de $\mathcal{E}-$longueur infinie.

Il est clair que les objets isomorphes possèdent la même longueur, ce qui nous permet de définir la fonction de longueur relative sur l'ensemble des classes d'isomorphismes des objets de la catégorie additive. Après, on montre que c'est une mesure au sens de Gabriel [Gab73]. On étudie les propriétés de cette longueur et on les compare aux propriétés satisfaites par le cas particulier et déjà connu de la fonction de longueur unique définie sur une catégorie abélienne. Cette nouvelle longueur ne satisfait pas exactement aux mêmes propriétés, mais elle possède quand même des caractéristiques intéréssantes, qui nous ont permis de prouver le résultat suivant :

Lemme 1.4. Un objet $\mathcal{E}-$fini X de $(\mathcal{A}, \mathcal{E})$ est $\mathcal{E}-$artinien et $\mathcal{E}-$noetherien (au sens de [BHLR18, Définition 6.4]).

Comme réponse à la question posée au début, on obtient le résultat suivant :

Théorème 1.5. Soient $\mathcal{E}$ et $\mathcal{E}'$ deux structures exactes sur $\mathcal{A}$ telles que $\mathcal{E}' \subseteq \mathcal{E}$. Alors $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ pour tout objet X de $\mathcal{A}$.

L'une des applications de cette fonction de longueur est qu'elle permet de généraliser la mesure de Gabriel-Roiter en adaptant sa définition originale comme suit :

Définition 1.6. Soit $ind\mathcal{A}$ la classe de toutes les classes d'isomorphisme d'objets indécomposables d'une catégorie additive $\mathcal{A}$. Une application $\mu_{\mathcal{E}} : (ind\mathcal{A}, \subset_{\mathcal{E}}) \to (\mathcal{P}, \leq)$ est appelée une mesure de Gabriel-Roiter sur une catégorie exacte $(\mathcal{A}, \mathcal{E})$ si elle satisfait les axiomes suivants :

(GR_1) $\mu_{\mathcal{E}}$ est une mesure

(GR_2) $\mu_{\mathcal{E}}(X) = \mu_{\mathcal{E}}(Y)$ implique que $l_{\mathcal{E}}(X) = l_{\mathcal{E}}(Y)$ pour tout $X, Y \in ind\,\mathcal{A}$

(GR_3) Si $l_{\mathcal{E}}(X) \geq l_{\mathcal{E}}(Y)$ et $\mu_{\mathcal{E}}(X') \lneqq \mu_{\mathcal{E}}(Y)$ pour tout $X' \subsetneq_{\mathcal{E}} X$, alors

$$\mu_{\mathcal{E}}(X) \leq \mu_{\mathcal{E}}(Y).$$

Pour assurer l'existence d'une telle fonction, on propose la description explicite suivante et on prouve qu'elle satisfait aux axiomes de la définition précédente.

Définition 1.7. On définit l'application

$$\mu_{\mathcal{E}} : (ind\mathcal{A}, \subset_{\mathcal{E}}) \to (\mathfrak{S}(\mathbb{N}), \lll)$$

par

$$X \longmapsto \mu_{\mathcal{E}}(X) = \max_{F_{\mathcal{E}}(X)} (l_{\mathcal{E}}(F_{\mathcal{E}}(X)))$$

en considérant la longueur maximale sur toutes les $F_{\mathcal{E}}(X)$. Ici les $\mathcal{E}$−filtrations $F_{\mathcal{E}}(X)$ d'un objet X

$$F_{\mathcal{E}}(X) = X_1 \subsetneq_{\mathcal{E}} \ldots \subsetneq_{\mathcal{E}} X_n = X$$

sont telles que tous les $\mathcal{E}$−sous-objets X_i sont indécomposables. On note le vecteur de longueur d'une telle filtrattion :

$$l_{\mathcal{E}}(F_{\mathcal{E}}(X)) = (l_{\mathcal{E}}(X_1), \ldots, l_{\mathcal{E}}(X)).$$

Ici $(\mathfrak{S}(\mathbb{N}), \lll)$ est l'ensemble des suites finies de nombres naturels muni de l'ordre lexicographique inversé.

On étudie cette mesure et on montre que la majorité des propriétés étudiées par H. Krause, pour le cas abélien dans [Kr07] restent valides.

Commentaires sur la contribution de chaque auteure ou auteur :

Cet article représente mon premier projet de recherche au doctorat. Je suis l'auteure principale de cet article et je suis responsable des sections suivantes : 1, 2, 3.1, 4.2, 4.3, 5, 6, 7 et 8.
Le Pr. Th. Brüstle a contribué comme étant le superviseur du projet, pendant que les étudiants Denis Langford et Sunny Roy ont contribué dans les calculs préliminaires des exemples de la page 13 de l'article.

Cet article est publié dans le Journal suivant : *Journal of Pure and Applied Algebra*.

CHAPITRE 2

L'intersection, la somme et la propriété de Jordan-Hölder pour les catégories exactes

Cet article est écrit en anglais dans sa version originale. Il est accessible dans l'annexe B de cette thèse. Écrit par Th.Brüstle, A.Tattar et moi même, cet article est publié dans le Journal suivant : *Journal of Pure and Applied Algebra*.

Le but de ce projet est de donner une version du théorème populaire de Jordan-Hölder-Schreier. Ce théorème dit que toutes les séries de composition d'un groupe sont équivalentes, c'est-à-dire qu'elles possèdent la même longueur et les mêmes facteurs de composition à permutation près. Ce résultat important est valide pour toute catégorie abélienne (voir [Po]) et notre but est d'étudier la possibilité de l'étendre aux catégories exactes.
On commence par généraliser les notions nécessaires à notre contexte. On adapte alors la définition d'une suite de composition en remplaçant les sous-objets (simples) et les inclu-

sions par des sous-objets admissibles ($\mathcal{E}-$simples) et des monomorphismes admissibles. Puis on considère la propriété de Jordan-Hölder relative à une structure exacte $\mathcal{E}$.
Les questions posées sont les suivantes :

Question 2.1. Est ce que la propriété relative de Jordan-Hölder est toujours satisfaite par les catégories exactes ? Quand est-ce qu'une catégorie exacte $(\mathcal{A}, \mathcal{E})$ satisfait à cette propriété ? Existe-t-il des exemples de catégories exactes de Jordan-Hölder autres que les catégories abéliennes munies de leurs structures maximales ?

Étant donné que la réponse n'est ni évidente, ni immédiate, cette question devient intéressante. On aimerait savoir pour quelles catégories exactes la propriété est satisfaite et de comprendre mieux quand cela arrive.
Pour répondre à ces questions, on choisit de commencer par étudier les sous-objets admissibles. Notre approche est de commencer par généraliser les notions d'intersection et de somme de sous-objets admissibles.
En premier lieu, on adapte ces notions comme définies pour les catégories abéliennes, en utilisant le produit fibré et la somme amalgamée. Notons que cette intersection et cette somme ne sont pas toujours des sous-objets admissibles de l'objet initial. Alors, pour assurer l'admissibilité, on considère les catégories (AI), (AS) et (AIS). Ces catégories exactes sont définies par moi même et S. Roy dans un travail antécédant [HR19]. Comme les définitions suivantes le montrent, elles forment des sous-classes des catégories exactes de Quillen :
On appelle catégories exactes nécessairement stables pour les **I**ntersections **A**dmissibles, dans le sens introduit là-dessous, les **A.I**-catégories :

Définition 2.2. [HR19, Définition 4.3]Une catégorie exacte $(\mathcal{A}, \mathcal{E})$ est appelée une *AI-catégorie* si $\mathcal{A}$ est une catégorie pré-abélienne satisfaisant à l'axiome additionnel suivant :

(AI) Le produit fibré A de deux monomorphismes admissibles $j : C \rightarrowtail D$ et $g : B \rightarrowtail D$ existe et ses deux morphismes canoniques sont des monomorphismes admissibles i

et f :

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, tail, "f"'] & B \arrow[d, tail, "g"] \\ C \arrow[r, tail, "j"'] & D \arrow[ul, phantom, very near start, "\lrcorner"] \end{tikzcd}$$

Ces catégories ont été récemment citées par les auteurs de [BCKT21], qui les ont utilisées pour généraliser les théorèmes sur les décompositions uniformes uniques des catégories des modules.

On rappelle maintenant la définition d'une sous-classe spéciale des AI-catégories, qu'on appelle les **A.I.S** catégories exactes, vu qu'elles admettent les **I**ntersections et les **S**ommes **A**dmissibles :

Définition 2.3. [HR19, Définition 4.5] Une catégorie exacte $(\mathcal{A}, \mathcal{E})$ est dite une *AIS-catégorie* si c'est une AI-catégorie satisfaisant à l'axiome additionnel suivant :

(AS) Le morphisme u dans le diagramme ci-dessous, donné par la propriété universelle de la somme amalgamée de i et f, est un monomorphisme admissible :

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, tail, "f"'] \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[d, tail, "l"] \arrow[ddr, tail, bend left, "g"] & \\ C \arrow[r, tail, "k"'] \arrow[drr, tail, bend right, "j"'] & E \arrow[dr, tail, "u"] & \\ & & D \end{tikzcd}$$

J'ai étudié les catégories (AI) avec Th. Brüstle, A. Shah, A. Tattar et S-A. Wegner, dans [BHT20] et [HSW20], et nous avons plusieurs résultats dont la caractérisation suivante :

Théorème 2.4. Une catégorie exacte $(\mathcal{A}, \mathcal{E})$ est de type (AI) si et seulement si $\mathcal{A}$ est quasi-abélienne et $\mathcal{E} = \mathcal{E}_{all} = \mathcal{E}_{max}$.

Dans le but de bien comprendre la prochaine caractérisation, on rappelle la définition suivante :

Définition 2.5. Un morphisme $f : A \to B$ dans une catégorie exacte est dit *admissible* s'il se factorise comme $f = m \circ e$ avec m un monomorphisme admissible et e un épimorphisme admissible. On note un morphisme admissible

$$A \xrightarrow{f} \circ\!\!\rightarrow B$$

et $Hom^{ad}_{\mathcal{A}}(-,-)$ la classe de tous les morphismes admissibles dans $\mathcal{A}$.

On étudie les catégories (AIS) et on obtient une nouvelle caractérisation des catégories abéliennes comme étant les catégories où les morphismes admissibles sont fermés sous l'addition et la composition :

Théorème 2.6. Soit $(\mathcal{A}, \mathcal{E})$ une catégorie exacte. Alors les propriétés suivantes sont équivalentes :

a) $\mathcal{A}$ est une catégorie abélienne et $\mathcal{E} = \mathcal{E}_{max}$,

b) $(\mathcal{A}, \mathcal{E})$ est une AIS-catégorie,

c) $Hom(\mathcal{A}) = Hom^{ad}(\mathcal{A})$,

d) $Hom^{ad}(\mathcal{A})$ est stable pour la composition,

e) $Hom^{ad}(\mathcal{A})$ est stable pour l'addition,

où $Hom(\mathcal{A})$ est la classe de tous les morphismes de $\mathcal{A}$ et $Hom^{ad}(\mathcal{A})$ est la classe de tous les morphismes admissibles de $(\mathcal{A}, \mathcal{E})$ (voir [Bü, Définition 8.1]).

Suivant ces observations, on généralise dans [BHT20, Définition 5.5] les notions d'intersection et de somme pour toute catégorie exacte, sans aucune condition supplémentaire sur la catégorie additive. Celles-ci nous permettent de définir le radical de Jacobson [BHT20, Définition 6.1] et les catégories d'Artin-Wedderburn dans [BHT20, Définition 6.4].
Ainsi, on prouve une version du théorème de Jordan-Hölder de la façon suivante :

Théorème 2.7. Soit $(\mathcal{A}, \mathcal{E})$ une catégorie $\mathcal{E}$-Artin-Wedderburn exacte (dans le sens de [BHT20, Définition 6.4]). Alors $(\mathcal{A}, \mathcal{E})$ est une catégorie exacte de Jordan-Hölder (comme définit dans 0.12).

Pour les catégories de modules sur une algèbre de Nakayama, on obtient la caractérisation suivante :

Théorème 2.8. Soit Λ une algèbre de Nakayama et soit $\mathcal{A} = \operatorname{mod}\Lambda$ la catégorie des modules sur cette algèbre. Alors une catégorie exacte $(\mathcal{A}, \mathcal{E})$ est de $\mathcal{E}$–Artin-Wedderburn précisément quand elle est une catégorie exacte de Jordan-Hölder.

Finalement, on généralise la fonction de longueur de Jordan-Hölder et on étudie ses propriétés. Cette fonction de longueur est correctement définie seulement sur les catégories exactes de Jordan-Hölder et constitue un cas particulier de la longueur qu'on a définie pour toute catégorie exacte dans [BHLR18].

On note qu'on a aussi généralisé le quatrième théorème d'isomorphisme en prouvant la version relative à une structure exacte sur une catégorie additive à [BHT20, Proposition 3.8]. Ce théorème est nécessaire pour plusieurs de nos preuves.

Commentaires sur la contribution de chaque auteure ou auteur :

Cet article est le fruit de ma collaboration avec Pr. Thomas Brüstle et M. Aran Tattar. Les résultats obtenus sont dûs à des longues discussions et échanges d'idées, et donc, par conséquent, les contributions sont également distribuées entre les trois auteurs.
La contribution de M. Aran Tattar a été apportée pendant sa visite à l'Université de Sherbrooke durant le mois de Mars de l'année 2020.

Cet article est publié dans le Journal suivant : *Journal of Pure and Applied Algebra*.

CHAPITRE 3

Les treillis des structures exactes et des structures partiellement exactes

Cet article est écrit en anglais dans sa version originale, et est accessible dans l'annexe C de cette thèse. Écrit par Th.Brüstle, R.Baillargeon, M.Grosky et moi même, cet article est soumis pour publication au Journal suivant : *Journal of Algebra*.

Dans cet article, on considère une généralisation des structures exactes qu'on appelle *les structures partiellement exactes* :

Définition 3.1. Soit $\mathcal{A}$ une catégorie additive. Une *structure partiellement exacte* sur $\mathcal{A}$ est une classe de paires de noyaux-conoyaux (i, d) dans $\mathcal{A}$ qui est stable pour les isomorphismes et doit aussi être stable pour les **sommes directes**, satisfaisant seulement aux quatre axiomes $(E0)$, $(E0)^{op}$,$(E2)$ et $(E2)^{op}$ de la définition d'une structure exacte de Quillen.

On considère l'ensemble $Wex(\mathcal{A})$ de toutes les structures partiellement exactes sur une catégorie additive $\mathcal{A}$, partiellement ordonné par l'inclusion, et on prouve le résultat suivant :

Théorème 3.2. Soit $\mathcal{A}$ une catégorie additive et $\mathcal{E}_{max}$ la structure exacte maximale sur $\mathcal{A}$. Alors les structures partiellement exactes inclues dans $\mathcal{E}_{max}$ forment un treillis complet et borné

$$(Wex(\mathcal{E}_{max}), \subseteq, \wedge, \vee_W)$$

où la borne inférieure est donnée par l'intersection des classes et la borne supérieure $\vee_W$ est donnée par

$$\mathcal{W} \vee_W \mathcal{W}' = \cap\{\mathcal{V} \in Wex(\mathcal{A}) \mid \mathcal{W} \subseteq \mathcal{V}, \mathcal{W}' \subseteq \mathcal{V}\}.$$

Étant un sous-ensemble de $Wex(\mathcal{A})$, on se demande si le treillis des structures exactes $Ex(\mathcal{A})$, qu'on a introduit dans [BHLR18], forme aussi un sous-treillis du treillis $Wex(\mathcal{A})$. On conclut que ces deux treillis ne possèdent pas la même structure de treillis, plus précisément, leurs bornes supérieures sont différentes. Alors $Ex(\mathcal{A})$ est juste un sous-ensemble de $Wex(\mathcal{A})$, mais ne forme pas un sous-treillis de $Wex(\mathcal{A})$.

La relation entre les structures exactes et les sous-bifoncteurs additifs fermés (dans le sens de Butler et Horrocks [BuHo61]) du premier foncteur d'extension $Ext^1_{\mathcal{A}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \to Ab$, discutée par Yoneda depuis 1954 dans [Y54], a aussi été étudiée dans le travail suivant [DRSS99]. On initie une étude de ces sous-bifoncteurs et on utilise le résultat de W. Rump sur l'existence de $\mathcal{E}_{max}$ [Ru11], postérieur à [DRSS99], pour construire nos structures de treillis. Inspirés par l'idée de Yoneda, nous associons à chaque structure exacte $\mathcal{E}$ un sous-bifoncteur additif *fermé* $Ext^1_{\mathcal{E}}$ de $Ext^1_{\mathcal{E}_{max}}$. Puis, on munit l'ensemble $CBiFun(\mathcal{A})$ de tous les sous-bifoncteurs fermés du foncteur d'extension $Ext^1_{\mathcal{A}}$ d'une structure de treillis. Ceci nous permet d'établir un isomorphisme entre ces deux treillis et de conclure que $CBiFun(\mathcal{A})$ est aussi un treillis borné et complet.

D'un autre côté, on associe aussi à chaque structure partiellement exacte $\mathcal{W}$ un sous-bifoncteur additif de $Ext^1_{\mathcal{W}}$, et on établit un isomorphisme de treillis entre $Wex(\mathcal{A})$ et le treillis $BiFun(\mathcal{A})$ formé par tous les sous-bifoncteurs additifs du bifoncteur $Ext^1_{\mathcal{A}}$.
Notons que $CBiFun(\mathcal{A})$ ne forme pas un sous-treillis du treillis $BiFun(\mathcal{A})$.

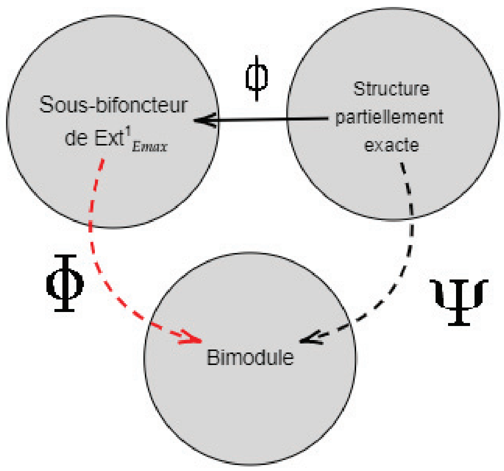
Inspirées par les structures exactes à droite et à gauche de [BC12], on définit les structures partiellement exactes à droite et à gauche et on prouve que toute structure partiellement exacte est construite à partir d'une paire de structures à droite et à gauche.

Afin de mieux étudier toutes ces structures, on propose d'aller chercher des nouveaux liens avec d'autres structures, ce qui permet de découvrir des nouvelles façons pour représenter tous ces treillis. Sous certaine conditions de finitude, on fournit une troisième façon beaucoup plus simple et accessible d'étudier ces treillis par un treillis de bimodules sur l'algèbre d'Auslander B de la catégorie $\mathcal{A}$, qu'on note $Bim(B)$. On caractérise les sous-bimodules (du bimodule maximal) associés aux sous-bifoncteurs fermés et on les appelle des sous-bimodules fermés. On démontre que ces sous-bimodules fermés $CBim(B)$ forment un treillis isomorphe à $Ex(\mathcal{A})$ et à $CBiFun(\mathcal{A})$ en même temps.
De plus, on étudie les liens entre les trois treillis isomorphes $Wex(\mathcal{A})$, $BiFun(\mathcal{A})$ et $Bim(B)$ et le treillis des sous-catégories topologiques de la catégorie des défauts notée def (voir [BBGH20, Section 6]). On obtient le résumé suivant :

Corollaire 3.3. Soit $\mathcal{A}$ une catégorie idempotente complète essentiellement petite, alors les treillis suivants sont isomorphes :

$$Wex(\mathcal{A}) \tilde{\to} BiFun(\mathcal{A}) \tilde{\to} Bim(B) \tilde{\to} def(\mathcal{E}^{\max}).$$

La figure suivante décrit brièvement les isomorphismes de treillis construits entre les différents treillis :



Intéressés par ces nouvelles structures partiellement exactes, nous étudions leurs propriétés et généralise plusieurs des résultats valides pour les catégories exactes, comme :

Théorème 3.4. (L'axiome obscur de Quillen pour les catégories *partiellement* exactes) Soit $\mathcal{W}$ une structure partiellement exacte sur une catégorie additive $\mathcal{A}$.

1. on considère les morphismes $A \xrightarrow{i} B \xrightarrow{j} C$ de $\mathcal{A}$, où i admet un conoyau et $j \circ i$ est un monomorphisme admissible de $\mathcal{W}$. Alors i est aussi un monomorphisme admissible de $\mathcal{W}$.
2. on considère les morphismes $X \xrightarrow{f} Y \xrightarrow{g} Z$ de $\mathcal{A}$, où g admet un noyau et $g \circ f$ est un épimorphisme admissible de $\mathcal{W}$. Alors g est aussi un épimorphisme admissible de $\mathcal{W}$.

L'un des résultats les plus intéressants généralisés est la caractérisation de la structure exacte maximale stable dûe à S. Crivei en 2012 dans [Cr19] après [Cr11]. Comme conséquence, on obtient le résultat suivant :

Corollaire 3.5. Pour toute catégorie additive idempotente complète, il existe une unique structure partiellement exacte maximale constituée par la classe de toutes les suites exactes courtes stables :

$$\mathcal{W}_{max} = \mathcal{E}_{max} = \mathcal{E}_{sta}.$$

Plus généralement, on introduit les catégories partiellement extriangulées qui généralisent *simultanément* les catégories extriangulées introduites dans [NP19] et les catégories partiellement exactes qu'on vient d'introduire. Puis on caractérise toutes les structures partiellement exactes parmi elles.

On considère aussi le treillis des sous-structures partiellement extriangulées et on établit un isomorphisme de treillis avec le treillis des sous-catégories topologisées.

Commentaires sur la contribution de chaque auteure ou auteur :

Pr. Thomas Brüstle, son étudiante Rose-Line Baillargeon et moi-même, avons travaillé sur ce projet depuis l'année passée, et avons rédigé la majorité de cet article. La contribution de l'auteur M. Mikhail Gorsky se résume à la section 6, ajoutée au dernier stade du projet.

Cet article est soumis pour publication au journal suivant : *Journal of Algebra.*

CONCLUSION

Le type d'une certaine catégorie additive est souvent reconnu par les propriétés des suites exactes courtes formées par des objets et morphismes de cette catégorie. Ce type nous permet de déterminer les structures exactes possibles existantes sur cette catégorie. Et vice versa, l'existence d'une certaine structure exacte sur une catégorie additive nous apprend beaucoup sur la catégorie elle même. Par exemple, le fait que la classe de toutes les suites exactes courtes forme une structure exacte sur une catégorie additive, implique que cette catégorie est quasi-abélienne, et l'inverse est bien sûr vrai aussi.

Ce travail nous a appris qu'on peut aller même plus loin jusqu'à dire que les propriétés des *sous-objets admissibles* d'une catégorie additive sont fortement liées au type de cette catégorie. Cette affirmation est renforcée par notre étude de la propriété de Jordan-Hölder relative à une structure exacte donnée sur une catégorie additive. Cette propriété est elle-même donnée par l'équivalence des filtrations d'un objet par des sous-objets admissibles. Les propriétés de quelques sous-objets en particulier, notamment l'intersection et la somme des sous-objets d'un objet, nous permettent de caractériser les catégories quasi-abéliennes et abéliennes. Les propriétés du radical et la semi-simplicité des objets, définissant les catégories d'Artin-Wedderburn exactes, nous permettent de caractériser, sous certaines conditions, les catégories exactes de Jordan-Hölder. Cette caractérisation, [BHT20, Théorème 6.12] permet de trouver des nouveaux exemples de catégories exactes de Jordan-Hölder. À part certains exemples évidents, tels qu'une catégorie abélienne munie de sa structure exacte maximale ou une catégorie de krull-schmidt munie de sa structure exacte minimale, voici un nouvel exemple de telles catégories : soit $mod\Lambda$ pour Λ une algèbre de Nakayama. Toute classe sans torsion est une catégorie exacte d'Artin-Wedderburn. Alors, en vertu de notre caractérisation, toute classe sans torsion sur $mod\Lambda$ est une catégorie exacte de Jordan-Hölder.

Les catégories exactes de Jordan-Hölder possèdent des propriétés très intéressantes. Par exemple, la fonction de longueur de Jordan-Hölder satisfait à des propriétés supplémentaires à celles satisfaites par la fonction de longueur générale qu'on a définie sur toute catégorie exacte. Alors que cette dernière satisfait à l'inégalité suivante :

Théorème 3.6. [BHLR18, Théorème 6.7] Soit $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ une suite exacte courte formée d'objets $\mathcal{E}$−finis. Alors

$$l_{\mathcal{E}}(Y) \geq l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Z).$$

La longueur de Jordan-Hölder vérifie l'égalité du corollaire suivant :

Corollaire 3.7. [BHT20, Corollaire 7.2] Soit

$$X \rightarrowtail Z \twoheadrightarrow Y$$

une suite exacte courte admissible d'une catégorie exacte finie de Jordan-Hölder. Alors

$$l_{\mathcal{E}}(Z) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y).$$

On observe de façon similaire que l'existence d'une suite de composition finie pour un objet $\mathcal{E}$−artinien et $\mathcal{E}$−noetherien est garantie dans toute catégorie exacte :

Lemme 3.8. [BHLR18, Lemme 6.6][BHT20, Lemme 7.5] Si X est $\mathcal{E}$−artinien et $\mathcal{E}$−noetherien alors X admet une $\mathcal{E}$−suite de composition.

Par contre, la longueur de telles suites de compositions n'est pas nécessairement unique en générale, vu qu'il peut y exister plusieurs $\mathcal{E}-$suites de compositions pour le même objet. Alors, en considérant une catégorie exact de Jordan-Hölder, on améliore 1.4 et le résultat précédant 3.8 de la façon suivante :

Théorème 3.9. [BHT20, Théorème 7.6] Un objet X de $(\mathcal{A}, \mathcal{E}_{JH})$ est $\mathcal{E}-$fini si et seulement si X est $\mathcal{E}-$artinien et $\mathcal{E}-$noetherien.

La réduction d'une structure exacte $\mathcal{E}$ sur une catégorie additive $\mathcal{A}$, revient à choisir une sous-structure exacte $\mathcal{E}' \subseteq \mathcal{E}$. Le fait de réduire la classe de suites exactes courtes initiales, réduit le nombre de morphismes admissibles, de monomorphismes et d'épimorphismes admissibles, et par conséquence réduit le nombre de sous-objets admissibles de tout objet de la catégorie. Alors, plus on réduit la structure exacte $\mathcal{E}$, plus le nombre de $\mathcal{E}-$simples augmente. Enfin, tous les objets indécomposables de $\mathcal{A}$ sont des $\mathcal{E}_{min}-$simples.
La réduction d'une structure exacte influence par exemple les propriétés des suites de compositions et les filtrations admissibles d'un objet. C'est ainsi que nous avons obtenu nos résultats sur la réduction des fonctions de longueurs :

Théorème 3.10. [BHLR18, Lemme 8.1] Soient $\mathcal{E}$ and $\mathcal{E}'$ deux structures exactes sur une catégorie additive $\mathcal{A}$, tel que $\mathcal{E}'$ est obtenu par la réduction suivante $\mathcal{E}' \subseteq \mathcal{E}$, alors $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ pour tout objet X de $\mathcal{A}$.

Voici un exemple illustrant comment la fonction de longueur se réduit en chiffres :

Exemple 3.11. On reconsidère le cube $Ex(\mathcal{A})$ des structures exactes sur $\mathcal{A} = repQ$ pour $Q = A_3$ (pour la construction et les définitions de base de cette catégorie voir [ASS93]) et on considère la chaine de réduction suivante

$$\mathcal{E}_{min} \subseteq \mathcal{E}_{1,3,5} \subseteq \mathcal{E}_{max}$$

ce qui implique que

$$l_{\mathcal{E}_{min}}(I_2) = 1 < l_{\mathcal{E}_{1,3,5}}(I_2) = 2 < l_{\mathcal{E}_{max}}(I_2) = 3.$$

Le choix d'une structure exacte sur une catégorie additive permet d'étudier beaucoup de sujets, déjà étudiés pour les catégories abéliennes. En considérant une catégorie additive comme une catégorie exacte, les suites exactes courtes admissibles se comportent bien et satisfont à des propriétés intéressantes qu'elles héritent du cas abélien, comme par exemple les lemmes d'homologie. Par contre, malgré ce fait, on a croisé plusieurs résultats qui sont uniquement valides pour les catégories abéliennes. Dans ce cas, l'étude des catégories exactes nous a permis d'en apprendre plus sur certaines catégories additive spécifiques (quasi-abéliennes ou abéliennes par exemple) et de les caractériser en utilisant les notions relatives.

Il existe certaines propriétés des sous-objets admissibles qu'on perd en réduisant la structure exacte maximale, même pour une catégorie abélienne. Si par exemple on considère l'ensemble partiellement ordonné de tous les sous-objets admissibles d'un objet, ce dernier ne forme pas toujours un treillis pour une catégorie exacte en général, même si la catégorie est de Jordan-Hölder. Contrairement au cas abélien, cet ensemble admet toujours une structure de treillis dont l'infimun est donnée par l'intersection et le suprémum est donnée par la somme. Ce fait nous a inspiré pour introduire les notions générales d'intersection et de somme, données par certains ensembles de sous-objets admissibles et non par des objets uniques.

Un autre exemple est donné par la nouvelle caractérisation donnée aux catégories abéliennes en matière de morphismes admissibles. L'ensemble de tous les morphismes admissibles d'une catégorie abélienne avec sa structure exacte maximale forme un anneau, et ce dernier est stable pour la somme et la composition des morphismes admissibles uniquement pour les catégories abéliennes.

Durant ce travail, on a réalisé que les structures partiellement exactes ne sont pas moins intéressantes que les structures exactes. Comme le montre ce travail, certaines des propriétés des structures exactes restent valides. Notamment, on a pu prouver l'existence d'une unique structure partiellement exacte maximale dans un cadre plutôt général, qu'est celui des catégories partiellement idempotentes complètes.

En comparant les treillis des structures exactes et celui des structures partiellement exactes, il est important de noter que l'ensemble partiellement ordonné des structures partiellement exactes est un sous-ensemble de celui des structures exactes, mais ne forme pas un sous-treillis et les deux structures de treillis sont différentes.

L'introduction des structures partiellement exactes et partiellement extriangulées, et de leurs treillis, ouvre la porte pour étudier beaucoup de notions et de résultats établis pour des catégories abéliennes ou exactes, voir même extriangulées.

Enfin, je trouve intéressant d'étudier les propriétés des treillis introduites. Par exemple, on sait que les treillis des structures exactes ou partiellement exactes ne sont pas toujours distributifs, mais on ne sait pas encore quand est-ce qu'ils satisfont à cette condition, et que veut dire exactement satisfaire à cette condition. L'une des questions ouvertes est de trouver une condition suffisante, sur une catégorie additive générale, pour que le treillis $(Ex(\mathcal{A}), \subseteq, \bigwedge, \bigvee)$ soit distributif. C'est-à-dire que pour toutes structures exactes $\mathcal{E}, \mathcal{E}', \mathcal{E}' \in Ex(\mathcal{A})$ les égalités suivantes sont satisfaites :

$$\mathcal{E} \bigvee (\mathcal{E}' \bigwedge \mathcal{E}'') = (\mathcal{E} \bigvee \mathcal{E}') \bigwedge (\mathcal{E} \bigvee \mathcal{E}'')$$

$$\mathcal{E} \bigwedge (\mathcal{E}' \bigvee \mathcal{E}'') = (\mathcal{E} \bigwedge \mathcal{E}') \bigvee (\mathcal{E}' \bigwedge \mathcal{E}'').$$

ANNEXE A

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Reduction of exact structures

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ABSTRACT

Examples of exact categories in representation theory are given by the category of Δ-filtered modules over quasi-hereditary algebras, but also by various categories related to matrix problems, such as poset representations or representations of bocses. Motivated by the matrix problem background, we study in this article the reduction of exact structures, and consider the poset $(Ex(\mathcal{A}), \subset)$ of all exact structures on a fixed additive category $\mathcal{A}$. This poset turns out to be a complete lattice, and under suitable conditions results of Enomoto's imply that it is boolean. We initiate in this article a detailed study of exact structures $\mathcal{E}$ by generalizing notions from abelian categories such as the length of an object relative to $\mathcal{E}$ and the quiver of an exact category $(\mathcal{A}, \mathcal{E})$. We investigate the Gabriel-Roiter measure for $(\mathcal{A}, \mathcal{E})$, and further study how these notions change when the exact structure varies.

1. Introduction

There are several notions of exact categories given by Barr, Buchsbaum or Quillen. We study in this article Quillen's [18] notion of exact category, which is formulated in the context of an additive category $\mathcal{A}$. One specifies a distinguished class $\mathcal{E}$ of short exact sequences which forms an exact structure on $\mathcal{A}$, that is, $\mathcal{E}$ consists of kernel-cokernel pairs subject to some closure requirements, see section 2. The pair $(\mathcal{A}, \mathcal{E})$ is called an exact category (we also refer to [12] and [6] for the system of axioms we are using).

It is well known that on every additive category $\mathcal{A}$ the class of all split exact sequences provides the smallest exact structure, see [6, Lemma 2.7]. However, for the maximal exact structure there is quite some recent literature, such as [31], [8], [27] and [28] which shows that every additive category admits a unique maximal exact structure $\mathcal{E}_{max}$. We recall the details in section 2.

Quillen defined the abstract notion of exact structure somewhat as a by-product in his fundamental paper on higher algebraic K-theory. It allows to perform homological algebra relative to the exact structure $\mathcal{E}$, and to study the (relative) Grothendieck group and the derived category of $(\mathcal{A}, \mathcal{E})$, see [6]. Relative homological algebra (like relative projective objects) has also been studied intensely from a different point of view, starting with a paper by Auslander and Solberg [2] where they look at subbifunctors of the Ext-functor. It

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has been shown in [9] that these two concepts coincide, that is, the additive closed subbifunctors correspond to exact structures. Recently, exact structures have become focus of work by several authors, like [10] who classifies exact structures on a given Krull-Schmidt category of finite type, using Auslander algebra, or [14] where the more general concept of extriangulated structures is studied.

While every exact category $(\mathcal{A}, \mathcal{E})$ can be embedded into a module category, notions like length or simple object cannot be borrowed from such an embedding. The first goal of this paper is to give an intrinsic definition relative to the class of morphisms in $\mathcal{E}$, thus, in section 3, we call an object $\mathcal{E}$-simple if it does not admit proper monomorphisms that belong to the class $\mathcal{E}$. And we say that X is an $\mathcal{E}$-subobject of Y, or $X \subset_{\mathcal{E}} Y$ if there exists a monomorphism in $\mathcal{E}$ from X to Y. This change of definition requires to work out a number of notions and results that are granted in abelian categories, such as the notion of simple objects, artinian and finite objects or the length of an object. It turns out that in general not all the desired properties can be guaranteed. We also define, in section 3, the notion of the quiver of an exact category $Q(\mathcal{A}, \mathcal{E})$.

The motivation for studying reductions of exact structures stems from the matrix reduction technique. The method of matrix reduction has been applied successfully by the Kiev school to solve various important problems in representation theory, like the Brauer-Thrall conjectures, or to show the tame-wild dichotomy. While the basic technique is elementary, the formalism of matrix reductions is somewhat complicated. Various models have been proposed to formalize matrix reductions: poset representations or bimodule problems cover only some cases. For the general case, one needs to study bocs representations, as introduced by Roiter in [24], or iterated quotients of bimodule problems as in [4].

No matter which formalism one chooses, the iterated application of reductions leads to more and more complicated categories. We propose a different approach in this paper, that is: *Keep the objects of the original category, but change its exact structure.* We illustrate in section 4 with an example that the elementary technique of matrix reduction can be viewed as a reduction of exact structures, where we define the *reduction* of an exact category $(\mathcal{A}, \mathcal{E})$ as the choice of an exact structure $\mathcal{E}' \subseteq \mathcal{E}$. We observe that when $\mathcal{E}'$ is the smallest possible exact structure, the split exact structure, then the exact category $(\mathcal{A}, \mathcal{E}')$ is in some sense semisimple: Every indecomposable is simple. In general, $(\mathcal{A}, \mathcal{E}')$ will be "simpler" than $(\mathcal{A}, \mathcal{E})$ in the sense that $(\mathcal{A}, \mathcal{E}')$ will have more simple objects.

We like to mention that the category of poset or bocs representations admits a natural exact structure, but these cases are rather special: the exact categories stemming from bocses always admit sufficiently many projectives, and are hereditary (the higher Ext groups vanish). The reduction of exact structures studied in this paper is therefore more general.

A second goal of this paper is to study for a fixed additive category $\mathcal{A}$ the poset $(Ex(\mathcal{A}), \subset)$ of exact structures ordered by containment. It turns out that this poset is a complete bounded lattice, see section 5.

Another goal is to generalize the notion of Gabriel-Roiter measure to the realm of exact categories. To start, we first define in section 6 the length of an object in an exact category $(\mathcal{A}, \mathcal{E})$: the $\mathcal{E}$-length $l_E(X)$ of an object X is the maximal length of a chain of proper $\mathcal{E}$-subobjects of X. We use the notion of $\mathcal{E}$-length to show the following result:

Proposition 1.1. *(see 6.11): Let $(\mathcal{A}, \mathcal{E})$ be an essentially small exact category where every object has finite $\mathcal{E}$-length. Then the relation $\subset_{\mathcal{E}}$ induces a partial order on $Obj\,\mathcal{A}$.*

This result allows to show that the length function $l_{\mathcal{E}}$ of a finite essentially small exact category $(\mathcal{A}, \mathcal{E})$ is a measure for the poset $Obj\,\mathcal{A}$ in the sense of Gabriel [11]. We further show that *most* of the work of Krause [16] on the Gabriel-Roiter measure for abelian length categories can be generalized to the context of exact categories: For the partially ordered set $(ind\,\mathcal{A}, \subset_{\mathcal{E}})$ equipped with the length function $l_{\mathcal{E}}$, we define the Gabriel-Roiter measure as a morphism of partially ordered sets which refines the length function $l_{\mathcal{E}}$, see Theorem 7.7:

Theorem 1.2. *There exists a Gabriel-Roiter measure for* $ind(\mathcal{A}, \mathcal{E})$.

Finally, starting from the maximal exact structure on $\mathcal{A}$ one can choose a sequence of reductions to arrive at the minimal, the split exact structure. In 8 we study these chains of exact structures in the lattice $Ex(\mathcal{A})$ and how basic notions, like the extended notion of length of an object, change under these reductions:

Proposition 1.3. *If* $\mathcal{E}$ *and* $\mathcal{E}'$ *are exact structures on* $\mathcal{A}$*, such that* $\mathcal{E}' \subseteq \mathcal{E}$*, then* $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ *for all objects* X *of* $\mathcal{A}$.

Thus, the length of objects is reduced, until the reduction reaches the minimum $\mathcal{E}_{min}$, where every indecomposable has length one, that is, simple.

2. Definitions and basic properties

2.1. Quillen exact categories

We recall from [12,6] the definition of exact categories in the sense of Quillen [18] and give some examples.

Definition 2.1. Let $\mathcal{A}$ be an additive category. A kernel-cokernel pair (i, d) in $\mathcal{A}$ is a pair of composable morphisms such that i is kernel of d and d is cokernel of i. If a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathcal{A}$ is fixed, an *admissible monic* is a morphism i for which there exists a morphism d such that $(i, d) \in \mathcal{E}$. An *admissible epic* is defined dually. Note that admissible monics and admissible epics are referred to as inflation and deflation in [12], respectively. We depict an admissible monic by $\rightarrowtail$ and an admissible epic by $\twoheadrightarrow$. An *exact structure* $\mathcal{E}$ on $\mathcal{A}$ is a class of kernel-cokernel pairs (i, d) in $\mathcal{A}$ which is closed under isomorphisms and satisfies the following axioms:

- (E0) For all objects A in $\mathcal{A}$ the identity 1_A is an admissible monic
- $(\mathrm{E0})^{op}$ For all objects A in $\mathcal{A}$ the identity 1_A is an admissible epic
- (E1) The class of admissible monics is closed under composition
- $(\mathrm{E1})^{op}$ The class of admissible epics is closed under composition
- (E2) The push-out of an admissible monic $i : A \rightarrowtail B$ along an arbitrary morphism $t : A \to C$ exists and yields an admissible monic s_C:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "t"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "s_B"] \\ C \arrow[r, tail, "s_C"'] & S \end{tikzcd}$$

- $(\mathrm{E2})^{op}$ The pull-back of an admissible epic h along an arbitrary morphism t exists and yields an admissible epic p_B

$$\begin{tikzcd} P \arrow[r, two heads, "p_B"] \arrow[d, "P_A"'] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d, "t"] \\ A \arrow[r, two heads, "h"'] & C \end{tikzcd}$$

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ consisting of an additive category $\mathcal{A}$ and an exact structure $\mathcal{E}$ on $\mathcal{A}$. Elements of $\mathcal{E}$ are called short exact sequences. Note that $\mathcal{E}$ is an exact structure on $\mathcal{A}$ if and only if $\mathcal{E}^{op}$ is an exact structure on $\mathcal{A}^{op}$. For a fixed additive category $\mathcal{A}$, we denote by $\mathrm{Ex}(\mathcal{A})$ the poset of exact structures $\mathcal{E}$ on $\mathcal{A}$, with order relation given by containment. In fact, $\mathrm{Ex}(\mathcal{A})$ is a lattice, see section 5.

Example 2.2. [6, Lemma 10.20] Let $(\mathcal{A}, \mathcal{E})$ be an exact category and $\mathcal{B}$ a full subcategory which is closed under extensions, that is, for every short exact sequence

$$X \rightarrowtail Y \twoheadrightarrow Z$$

in $\mathcal{E}$ the object Y belongs to $\mathcal{B}$ if the endterms X and Z are objects of $\mathcal{B}$. Then the pairs of $\mathcal{E}$ with components in $\mathcal{B}$ form an exact structure on $\mathcal{B}$. For example a torsion class of an abelian category forms an exact category since it is an extension closed subcategory.

2.2. Types of additive categories

Certain properties of the underlying additive category $\mathcal{A}$ determine which exact structures can exist on $\mathcal{A}$. We recall here the definition of various types of additive categories, and of some classes of short exact sequences. We then discuss in 2.3 and 2.4 some consequences on the existence of exact structures.

We begin with a large class of additive categories, the *weakly idempotent complete* categories:

Definition 2.3. Following [6], we call an additive category $\mathcal{A}$ *weakly idempotent complete* (w.i.c.) if all retractions have kernels and all sections have cokernels. In fact, Bühler shows in [6, Lemma 7.1] that it is sufficient to have one of the two conditions. Moreover, in [6, Corollary 7.5] it is shown that $\mathcal{A}$ is weakly idempotent complete if and only if every retraction is an admissible epic for all exact structures on $\mathcal{A}$, and dually, every section is an admissible monic for all exact structures on $\mathcal{A}$.

Definition 2.4. An additive category $\mathcal{A}$ is *idempotent complete* (i.c) if every morphism $e : X \to X$ in $\mathcal{A}$ satisfying $e^2 = e$ has a kernel, or equivalently, a cokernel.

Definition 2.5. An additive category $\mathcal{A}$ is *semi-abelian* if it is *pre-abelian* (has kernels and cokernels) and the induced canonical map

$$\bar{f} : Coim f \to Im f$$

is a bimorphism, i.e., a monomorphism and an epimorphism.

Definition 2.6. [19, p. 524] A kernel (A, f) is called *semi-stable* if for every push-out square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, "s_B"] \\ C \arrow[r, "s_C"'] & S \end{tikzcd}$$

the morphism s_C is also a kernel. We define dually a *semi-stable* cokernel. A short exact sequence $A \overset{i}{\rightarrowtail} B \overset{d}{\twoheadrightarrow} C$ is said to be *stable* if i is a semi-stable kernel and d is a semi-stable cokernel. We denote by $\mathcal{E}_{sta}$ the class of all *stable* short exact sequences.

Definition 2.7. A morphism f is called *strict* if the canonical map $\bar{f}$ is an isomorphism. A short exact sequence $A \overset{i}{\rightarrowtail} B \overset{d}{\twoheadrightarrow} C$ is said *strict* if i is strict or d is strict. We denote by $\mathcal{E}_{str}$ the class of all strict short exact sequences.

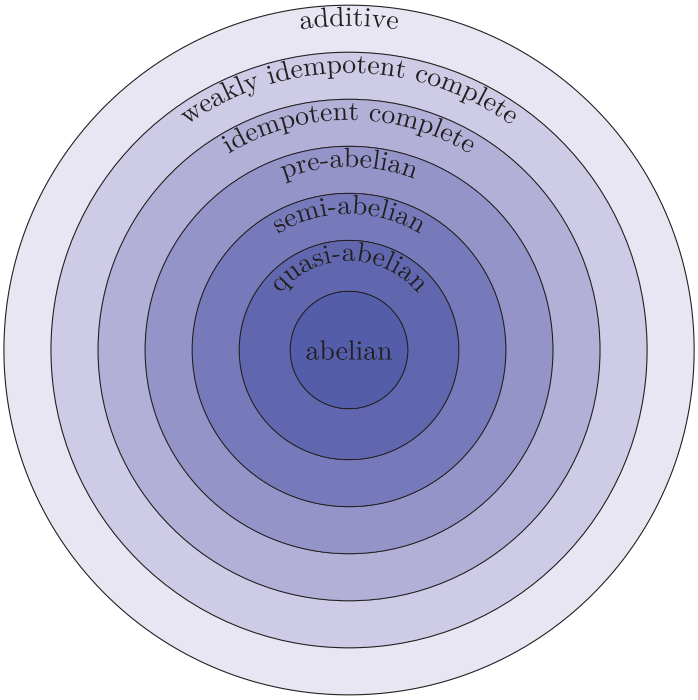
Definition 2.8. An additive category $\mathcal{A}$ is *quasi-abelian* if it is *pre-abelian* and all kernels and cokernels are *semi-stable*.

Moreover, an additive category $\mathcal{A}$ is *quasi-abelian* if it is *pre-abelian* and every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism.

Definition 2.9. An additive category $\mathcal{A}$ is *abelian* if it is *pre-abelian* and all morphisms are *strict*.

Remark 2.10. A *pre-abelian* category admits pullbacks and pushouts.

Remark 2.11. The hierarchy of additive categories which we discussed here is as follows (where all inclusions are strict):



2.3. The minimum exact structure

It is well known that every additive category admits a unique minimal exact structure $\mathcal{E}_{min}$:

Proposition 2.12. *[6, example 13.1] For every additive category $\mathcal{A}$ the sequences isomorphic to*

$$A \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} A \oplus B \xrightarrow{[01]} B$$

form an exact structure $\mathcal{E}_{min}$, called the split exact structure. In fact, every exact structure on $\mathcal{A}$ contains all split exact sequences [6, Lemma 2.7], which makes $\mathcal{E}_{min}$ the minimum in the lattice $Ex(\mathcal{A})$ of all exact structures on $\mathcal{A}$.

If $\mathcal{A}$ is weakly idempotent complete, then each retraction is isomorphic to a split sequence as above, hence the exact structure $\mathcal{E}_{min}$ is formed by all pairs (s, r) of sections with retractions.

Example 2.13. Let $S \subset (\mathbb{N}, +)$ be a submonoid, that is, S is an additively closed set containing zero. Consider the category $\mathcal{A}_S$ of vector spaces V over a field k of dimension $\dim_k V \in S$. For a short exact sequence in $\operatorname{mod} k$

$$0 \to U \to V \to W \to 0$$

we have that U and W in $\mathcal{A}_S$ implies V in $\mathcal{A}_S$ since S is additively closed. Thus $\mathcal{A}_S$ is additive since it is an extension-closed full subcategory of the additive category $\operatorname{mod} k$, and by Example 2.12, the split exact sequences in $\mathcal{A}_S$ form an exact structure $\mathcal{E}_{min}$ on $\mathcal{A}_S$. Note that $\mathcal{A}_S$ is not weakly idempotent complete when $S \neq \mathbb{N}$ since there are retractions whose kernel is not in $\mathcal{A}_S$.

2.4. The maximum exact structure

It is a deeper result that every additive category also admits a unique maximal exact structure $\mathcal{E}_{max}$. We review some of the recent literature on this subject:

Theorem 2.14. *[27, Corollary 2] Every additive category admits a unique maximal exact structure* $\mathcal{E}_{max}$.

The drawback of this result is that an explicit description of the maximum exact structure is not known in general. However, for certain types of additive categories, the exact structure $\mathcal{E}_{max}$ can be described explicitly. The following theorem generalizes the result on pre-abelian categories from [31, Theorem 3.3]:

Theorem 2.15. *[8, Theorem 3.5] Let $\mathcal{A}$ be an weakly idempotent complete category. Then the stable exact sequences $\mathcal{E}_{sta}$ define an exact structure on $\mathcal{A}$. Moreover, this is the maximal exact structure $\mathcal{E}_{max}$ on $\mathcal{A}$.*

Remark 2.16. The short exact sequences forming the maximal exact structure $\mathcal{E}_{max}$ do not always coincide with the stable short exact sequences in $\mathcal{E}_{sta}$. In fact we have that $\mathcal{E}_{max} \subseteq \mathcal{E}_{sta}$, so in case the class $\mathcal{E}_{sta}$ forms an exact structure it will be the maximal one. See [28] for an example where $\mathcal{E}_{max} \subsetneq \mathcal{E}_{sta}$.

Theorem 2.17. *[30, 1.1.7] ([6, 4.4]) In any quasi-abelian category, the class of all short exact sequences defines an exact structure $\mathcal{E}_{all}$ and this is the maximal one $\mathcal{E}_{max} = \mathcal{E}_{all}$. In particular this is the case for abelian categories (see also [25]).*

Remark 2.18. The class of all short exact sequences $\mathcal{E}_{all}$ does not necessarily form an exact structure for any additive category since pushouts of kernels need not be kernels. For a counter-example, take the category of abelian p-groups with no elements of infinite height, see [19, page 522]. But in case $\mathcal{E}_{all}$ forms an exact structure, it will be the maximal one.

2.5. More examples

Example 2.19. If $\mathcal{A}$ is a triangulated category then every monomorphism splits, and so $\mathcal{E}_{max} = \mathcal{E}_{sta} = \mathcal{E}_{min}$ forms the only possible exact structure on $\mathcal{A}$.

Example 2.20. [18] A quasi-abelian category $\mathcal{A}$ together with $\mathcal{E}_{str}$ is an exact category $(\mathcal{A}, \mathcal{E}_{str})$. See also [3, section 4].

Example 2.21. Every subcategory of an abelian category which is closed under direct sums and direct summands is idempotent complete and $\mathcal{E}_{max} = \mathcal{E}_{sta}$.

Example 2.22. [26] Let $A = kQ/I$ be the path algebra over a field k given by the following quiver Q with relations I generated by commutativity relations at the two squares (note that the algebra A is tilted of type E_6):

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] & 3 \arrow[l] \arrow[d] \\ 4 \arrow[r] & 5 & 6 \arrow[l] \end{tikzcd}$$

We consider the category $\mathcal{A} = A - proj$ of finitely generated projective A-modules. This $\mathcal{A}$ was the first example of a semi-abelian category which is not quasi-abelian. In particular, $\mathcal{A}$ is weakly idempotent complete and $\mathcal{E}_{max} = \mathcal{E}_{sta}$.

Example 2.23. Let $\mathcal{A}$ be the category of Banach spaces or Fréchet spaces, then $\mathcal{A}$ is quasi-abelian and $\mathcal{E}_{max} = \mathcal{E}_{all}$.

3. The quiver of an exact category

3.1. $\mathcal{E}$-subobjects and $\mathcal{E}$-simple objects

Throughout this section, let $\mathcal{E}$ be an exact structure for an additive category $\mathcal{A}$. We define the notion of $\mathcal{E}$-subobjects and $\mathcal{E}$-simple objects.

Definition 3.1. Let A and B be objects of $(\mathcal{A}, \mathcal{E})$. We write $A \subset_{\mathcal{E}} B$ and say that A is an *admissible subobject* or *$\mathcal{E}$-subobject of* B, if there is an admissible monic $A \overset{i}{\rightarrowtail} B$ from A to B. If in addition i is not an isomorphism, we use the notation $A \subsetneq_{\mathcal{E}} B$ and say that A is a *proper* admissible subobject of B.

Remark 3.2. An admissible monic $A \overset{i}{\rightarrowtail} B$ is proper precisely when its cokernel is non-zero. In fact, by uniqueness of kernels and cokernels, the exact sequence

$$B \overset{1_B}{\rightarrowtail} B \twoheadrightarrow 0$$

is, up to isomorphism, the only one with zero cokernel. Thus an admissible monic i has $\operatorname{coker} i = 0$ precisely when i is an isomorphism. Dually, an admissible epic $B \overset{d}{\twoheadrightarrow} C$ is an isomorphism precisely when $\ker d = 0$.

Definition 3.3. A non-zero object S in $(\mathcal{A}, \mathcal{E})$ is *$\mathcal{E}$-simple* if S admits no $\mathcal{E}$-subobjects except 0 and S, that is, whenever $A \subset_{\mathcal{E}} S$, then A is the zero object or isomorphic to S.

Remark 3.4. An $\mathcal{E}$-simple object is indecomposable, since the canonical inclusion $X_1 \overset{i_1}{\rightarrowtail} X_1 \oplus X_2$ is admissible in every exact structure, see Example 2.12. Conversely, when $\mathcal{E}$ is the split exact structure from Example 2.12, then every indecomposable object is $\mathcal{E}$-simple.

Example 3.5. Consider the category $\mathcal{A}_S$ of vector spaces from Example 2.13 for the monoid $S = \mathbb{N} \backslash \{1\}$, equipped with the split exact structure $\mathcal{E} = \mathcal{E}_{min}$. Then the $\mathcal{E}$-simple objects in $\mathcal{A}_S$ are k^2 and k^3, up to isomorphism. This corresponds to the fact that the monoid S admits $\{2, 3\}$ as minimal generating set.

3.2. The quiver of $(\mathcal{A}, \mathcal{E})$

The aim of this section is to define the quiver of an exact category, and compare it with different notions studied in the literature. We assume here that $\mathcal{A}$ is not only additive, but a k-category for some field k. It is shown in [9] that the datum of an exact structure $\mathcal{E}$ on $\mathcal{A}$ corresponds to the choice of an additive bifunctor

$$\mathrm{Ext}_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{mod}\, k$$

which is closed in the sense of M.C.R. Butler and G. Horrocks [7]. Here $\mathrm{Ext}_{\mathcal{E}}(Z, X)$ denotes the set of all exact pairs

$$X \xrightarrow{i} Y \xrightarrow{d} Z$$

in $\mathcal{E}$ modulo the usual equivalence relation of short exact sequences, which turns into a vector space under Baer sum.

Definition 3.6. We recall the (Jacobson) radical of an additive k-category and its powers from [1]:

– The radical $\mathrm{rad}_{\mathcal{A}}$ of an additive k-category $\mathcal{A}$ is the two-sided ideal given for all objects X and Y in $\mathcal{A}$ by the k-vector space $\mathrm{rad}_{\mathcal{A}}(X, Y)$ formed by all $f \in \mathcal{A}(X, Y)$ such that $1_X - g \circ f$ is invertible for all $g \in \mathcal{A}(Y, X)$.

– Given $m \geq 1$, the m^{th} power $rad^m_{\mathcal{A}} \subseteq rad_{\mathcal{A}}$ of $rad_{\mathcal{A}}$ is obtained by taking for $rad^m_{\mathcal{A}}(X, Y)$ the subspace of $rad_{\mathcal{A}}(X, Y)$ containing all finite sums of morphisms of the form

$$X = X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m = Y$$

where $f_i : X_{i-1} \to X_i \in rad_{\mathcal{A}}(X_{i-1}, X_i)$ for all $i = 1, \dots, m$.

Remark 3.7. If X and Y are indecomposable objects, then $\mathrm{rad}_{\mathcal{A}}(X, Y)$ is the k-vector space of all noninvertible morphism from X to Y.

Definition 3.8. The *quiver* $Q(\mathcal{A}, \mathcal{E})$ *of the exact category* $(\mathcal{A}, \mathcal{E})$ is the graded quiver given as follows:

– the vertex set $Q_0(\mathcal{A}, \mathcal{E})$ is the set of isomorphism classes of $\mathcal{E}$-simple objects.

For two vertices represented by $\mathcal{E}$-simple objects X and Y, we further define:

– the number of arrows of degree zero from X to Y equals the dimension of the space $\mathrm{irr}\,(X, Y) = \mathrm{rad}_{\mathcal{A}}(X, Y)/\mathrm{rad}^2_{\mathcal{A}}(X, Y)$ of irreducible morphisms in $\mathcal{A}$ from X to Y.

– the number of arrows of degree one from X to Y equals the dimension of the vector space $\mathrm{Ext}_{\mathcal{E}}(X, Y)$.

We draw in illustrations the arrows of degree zero by dotted lines, and the arrows of degree one by solid lines.

Example 3.9. Let A be an artinian k-algebra, and $\mathcal{A} = \mathrm{mod}\, A$. When $\mathcal{E} = \mathcal{E}_{all}$ is the maximal exact structure $\mathcal{E}_{max}$, then the quiver $Q(\mathcal{A}, \mathcal{E}_{max})$ is the ordinary (Gabriel) quiver of the algebra A, with all arrows of degree one. For the minimal exact structure $\mathcal{E} = \mathcal{E}_{min}$, the simples are the indecomposable A-modules by 3.4, and the quiver $Q(\mathcal{A}, \mathcal{E}_{min})$ is the Auslander-Reiten quiver of A, with all arrows of degree zero. We will discuss in section 4 how reduction of exact structures transforms iteratively the Gabriel quiver into the Auslander-Reiten quiver of an algebra.

Example 3.10. The technique of matrix reduction has been studied using various models, such as representations of posets, subspace categories, bimodules or bocses. We recall here from [12, 2.3 example 6] one example, the representations of posets (see also the books [20,29]): Given a poset $(S, \leq)$, the category $\operatorname{rep} S$ of representations of S is formed by matrices whose columns are subdivided into blocks corresponding to the elements $\{s_1, \ldots, s_n\}$ of S. More formally, the objects of $\operatorname{rep} S$ are pairs (d, M) where $d \in \mathbb{N}^{n+1}$, and M is a matrix with entries in k that has d_0 rows and $d_1 + \cdots + d_n$ columns, subdivided into n blocks of size $d_1, \ldots, d_n$:

$$M = \left[\, M_1 \middle| \cdots \middle| M_n \,\right]$$

A morphism $(d, M) \to (d', M')$ is given by a pair of matrices (X, Y) such that $XM = M'Y$ and where the matrix Y has a block structure determined by the order relation in S, allowing operations from columns in block i to columns in block j only if $s_i \leq s_j$ in S. One could equivalently define an element in rep S as a couple (V,M) where M is the same matrix as defined earlier and V is a set of $n+1$ k-vector spaces $\{V_0, V_1, ..., V_n\}$ of dimensions $\{d_0, d_1, ..., d_n\}$ respectively. Thus, a morphism can be illustrated with the following commutative diagram:

$$\begin{tikzcd} V_0 \arrow[d, "X"'] & V_1 \times ... \times V_n \arrow[l, "M"] \arrow[d, "Y"] \\ V_0' & V_1' \times ... \times V_n' \arrow[l, "M'"'] \end{tikzcd}$$

As in [12, section 9.1, example 5] (see also [9, 4.2] or [5, 2.3]), we equip the k-category $\mathcal{A} = \operatorname{rep} S$ with the exact structure $\mathcal{E}$ whose admissible monics are formed by morphisms (X, Y) where both X and Y are sections.

Let us consider (V,M) in rep S where $d_0 \geq 1$ and the following morphism:

$$s_0 : \quad \begin{tikzcd} k \arrow[d] & 0 \times ... \times 0 \arrow[l] \arrow[d] \\ V_0 & V_1 \times ... \times V_n \arrow[l, "M"'] \end{tikzcd}$$

The object $s_0 := (\{k, 0, ..., 0\}, 0)$ is a simple $\mathcal{E}$-subobject of (V, M). In fact s_0 is the unique simple object having $d_0 \geq 1$. Let us now fix $d_0 = 0$ and $(V, M) \neq (0, 0)$. Suppose $d_i \geq 1$ for a certain i, then the following morphism gives an $\mathcal{E}$-subobject s_i of (V,M):

$$s_i : \quad \begin{tikzcd} 0 \arrow[d] & 0 \times ... \times k \times ... \times 0 \arrow[l] \arrow[d, "q_i"] \\ 0 & V_1 \times ... \times k^{d_i} \times ... \times V_n \arrow[l] \end{tikzcd}$$

where $s_i = (V', 0)$ with $V_i' = k$ and all other spaces zero. This shows that the set $\{s_0, s_1, ..., s_n\}$ gives all $\mathcal{E}$-simple objects in rep S. There is a non-zero morphism f_{ij} from s_i to s_j whenever $s_i \leq s_j$ in the poset S. Note that each of these morphisms f_{ij} is a monomorphism and an epimorphism in the category $\operatorname{rep} S$, but for $i \neq j$ these are not isomorphisms, and not admissible monics nor admissible epics.

Furthermore, the following family of short exact sequences determines arrows of degree one in the quiver $Q(\mathcal{A},\mathcal{E})$:

$$\begin{tikzcd}
s_i : & 0 \arrow[d] & 0 \times \ldots \times k \times \ldots \times 0 \arrow[l] \arrow[d, "q_i"] \\
& k \arrow[d, "1"] & 0 \times \ldots \times k \times \ldots \times 0 \arrow[l, "1"'] \arrow[d] \\
s_0 : & k & 0 \times \ldots \times 0 \arrow[l]
\end{tikzcd}$$

In fact, one can verify that $\dim \operatorname{Ext}_{\mathcal{E}}(s_0, s_i) = 1$ and that there are no other extensions between these objects. The quiver $Q(\mathcal{A},\mathcal{E})$ is therefore formed by the Hasse quiver of S with arrows of degree zero, together with an extra vertex s_0 that sends an arrow of degree one to each vertex of S. We illustrate below an example of the quiver $Q(\mathcal{A},\mathcal{E})$ for a poset S with Hasse diagram $H(S)$:

$$H(S):\ \begin{tikzcd}
& s_1 \arrow[dl] \arrow[dr] & \\
s_2 \arrow[dr] & & s_3 \arrow[dl] \\
& s_4 &
\end{tikzcd}
\qquad
Q(\mathcal{A},\mathcal{E}):\ \begin{tikzcd}
& s_1 \arrow[dl, dotted] \arrow[ddr, dotted] & & \\
s_2 \arrow[ddr, dotted] & & & s_0 \arrow[ull] \arrow[lll] \arrow[dl] \arrow[ddll] \\
& & s_3 \arrow[dl, dotted] & \\
& s_4 & &
\end{tikzcd}$$

4. Matrix reduction versus exact structures

The aim of this section is to link matrix reduction to reduction of exact structures. We first present in 4.1 as an example a chain of matrix reductions, and illustrate some intermediate steps by certain quivers with dashed and solid arrows. We then justify these pictures in 4.2 and 4.3, showing that they are in fact the quivers of certain exact categories corresponding precisely to the intermediate steps of matrix reductions.

Definition 4.1. A reduction of an exact category $(\mathcal{A},\mathcal{E})$ is the choice of an exact structure $\mathcal{E}' \subseteq \mathcal{E}$ giving rise to a new exact category $(\mathcal{A},\mathcal{E}')$. Here we mean by $\mathcal{E}' \subseteq \mathcal{E}$ that every exact pair $(i,d) \in \mathcal{E}'$ also belongs to $\mathcal{E}$.

4.1. Matrix reduction

We describe here an example of a matrix reduction, and later compare it to reduction of exact structures. The matrix reduction is discussed in [12, 1.2], we refer to some background there.

Reduction for a quiver of type A_3 is also discussed in example 4.56 in [17], where the bocs point of view is given, compare the biquivers shown there. Consider the category $\mathcal{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

The category $\mathcal{A}$ is equivalent to the category $\operatorname{rep} S$ of representations of the poset $S = \{1,3\}$ of two incomparable elements. As in Example 3.10, its objects (d,M) are given by pairs of matrices A and B with the same number of rows, we write it as follows:

$$M = [A|B]$$

The algebra Λ operating on representations of dimension vector $d = (d_1,d_2,d_3)$ is given by pairs of square matrices (X,Y) where $X \in k^{d_2 \times d_2}$ and

$$Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_3 \end{bmatrix}.$$

Note that Λ is semisimple, and the quiver of Λ has three isolated vertices corresponding to the three simple representations S_1, S_2, S_3 of Q. The arrows of Q describe extensions between these simple objects, corresponding to the two matrices A and B.

(1) We choose in the first reduction step to transform the matrix A into its normal form:

$$[A|B] \longrightarrow \left[\begin{array}{cc|c} 0 & 1 & B' \\ 0 & 0 & B'' \end{array}\right]$$

Here we denote by 1 the identity matrix of size rank A. We now restrict the operating algebra to the subalgebra $\Lambda_1 \subset \Lambda$ formed by those pairs (X, Y) that preserve the normal form on the matrix A, that is,

$$X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 & 0 \\ 0 & Y_3 \end{bmatrix}.$$

Thus the matrix X is replaced by a matrix

$$X' = \begin{bmatrix} X_1' & X_{12} \\ 0 & X_1'' \end{bmatrix}$$

which induces a subdivision of the rows into two blocks labeled $2'$ and $2''$. The quiver of the algebra Λ_1 turns out to be the following:

$$2'' \dashrightarrow 2' \dashrightarrow 1 \qquad\qquad 3$$

The yet unreduced part of the matrix, given by the two blocks B' and B'', corresponds to extensions from 3 to the row-blocks $2'$ and $2''$. We might visualize this by introducing two solid arrows:

$$\begin{tikzcd} & 3 \arrow[dl, "\beta''"'] \arrow[d, "\beta'"] & \\ 2'' \arrow[r, dashed] & 2' \arrow[r, dashed] & 1 \end{tikzcd}$$

(2) In the second reduction step, we transform the matrix B'' into its normal form, and use row transformations to produce a zero block above the newly created identity matrix in the part corresponding to matrix B:

$$\left[\begin{array}{cc|c} 0 & 1 & B' \\ 0 & 0 & B'' \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 0 & 1 & B_3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(3) In the final reduction step, we transform the matrix B_3 to normal form:

$$\left[\begin{array}{cc|cc} 0 & 1 & B_3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

At this moment, the representation (d, M) is completely decomposed into a direct sum of indecomposable representations. The algebra Λ_3 of transformations preserving this decomposition is Morita-equivalent to the Auslander algebra of Q, hence the quiver of Λ_3 is the Auslander-Reiten quiver of Q.

4.2. Reduction of exact structures

To illustrate the change of exact structures, we consider in this section one example of an additive category $\mathcal{A}$ and describe all its exact structures. We recall from [6, 2.10] that an exact structure $\mathcal{E}$ can be viewed as an additive subcategory of the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes of objects in $\mathcal{A}$. Thus, we can talk about indecomposable short exact sequences, and use notation like the direct sum $e \oplus e'$ of short exact sequences in $\mathcal{E}$, or $\mathrm{add}(e)$ denoting the additive subcategory of $\mathcal{E}$ generated by the short exact sequence (e).

We reconsider now the following example

Example 4.2. Consider the category $\mathcal{A} = \mathrm{rep}\, Q$ of representations of the quiver

$$Q: \qquad 1 \xrightarrow{\ \alpha\ } 2 \xleftarrow{\ \beta\ } 3$$

The Auslander-Reiten quiver of $\mathcal{A}$ is as follows:

$$\begin{tikzcd}
 & P_1 \arrow[dr] \arrow[rr, dotted, no head] & & S_3 \\
S_2 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & I_2 \arrow[ur] \arrow[dr] & \\
 & P_3 \arrow[ur] \arrow[rr, dotted, no head] & & S_1
\end{tikzcd}$$

We consider the following indecomposable non-split exact sequences where the first three are the Auslander-Reiten sequences:

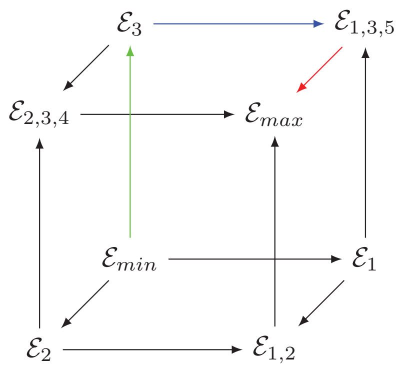
$$\begin{array}{rl}
(\mathrm{AR1}) & 0 \longrightarrow P_1 \longrightarrow I_2 \longrightarrow S_3 \longrightarrow 0 \\
(\mathrm{AR2}) & 0 \longrightarrow P_3 \longrightarrow I_2 \longrightarrow S_1 \longrightarrow 0 \\
(\mathrm{AR3}) & 0 \longrightarrow S_2 \longrightarrow P_1 \oplus P_3 \longrightarrow I_2 \longrightarrow 0 \\
(4) & 0 \longrightarrow S_2 \longrightarrow P_1 \longrightarrow S_1 \longrightarrow 0 \\
(5) & 0 \longrightarrow S_2 \longrightarrow P_3 \longrightarrow S_3 \longrightarrow 0
\end{array}$$

The following list enumerates *all* exact structures $\mathcal{E}$ on $\mathcal{A}$:

- $\mathcal{E}_{min}$ is the class of all split short exact sequences,
- $\mathcal{E}_1 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in \mathrm{add}\,(AR1)\}$,
- $\mathcal{E}_2 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in \mathrm{add}\,(AR2)\}$,
- $\mathcal{E}_3 = \{X \oplus Y | X \in \mathcal{E}_{min}, Y \in \mathrm{add}\,(AR3)\}$,
- $\mathcal{E}_{1,2} = \{X \oplus Y \ | X \in \mathcal{E}_1, Y \in \mathcal{E}_2\}$,
- $\mathcal{E}_{1,3,5} = \{X \oplus Y \oplus Z | X \in \mathcal{E}_1, Y \in \mathcal{E}_3, Z \in \mathrm{add}\,(5)\}$,
- $\mathcal{E}_{2,3,4} = \{X \oplus Y \oplus Z | X \in \mathcal{E}_2, Y \in \mathcal{E}_3, Z \in \mathrm{add}\,(4)\}$,
- $\mathcal{E}_{max}$ is the class of all short exact sequences in $\mathcal{A}$.

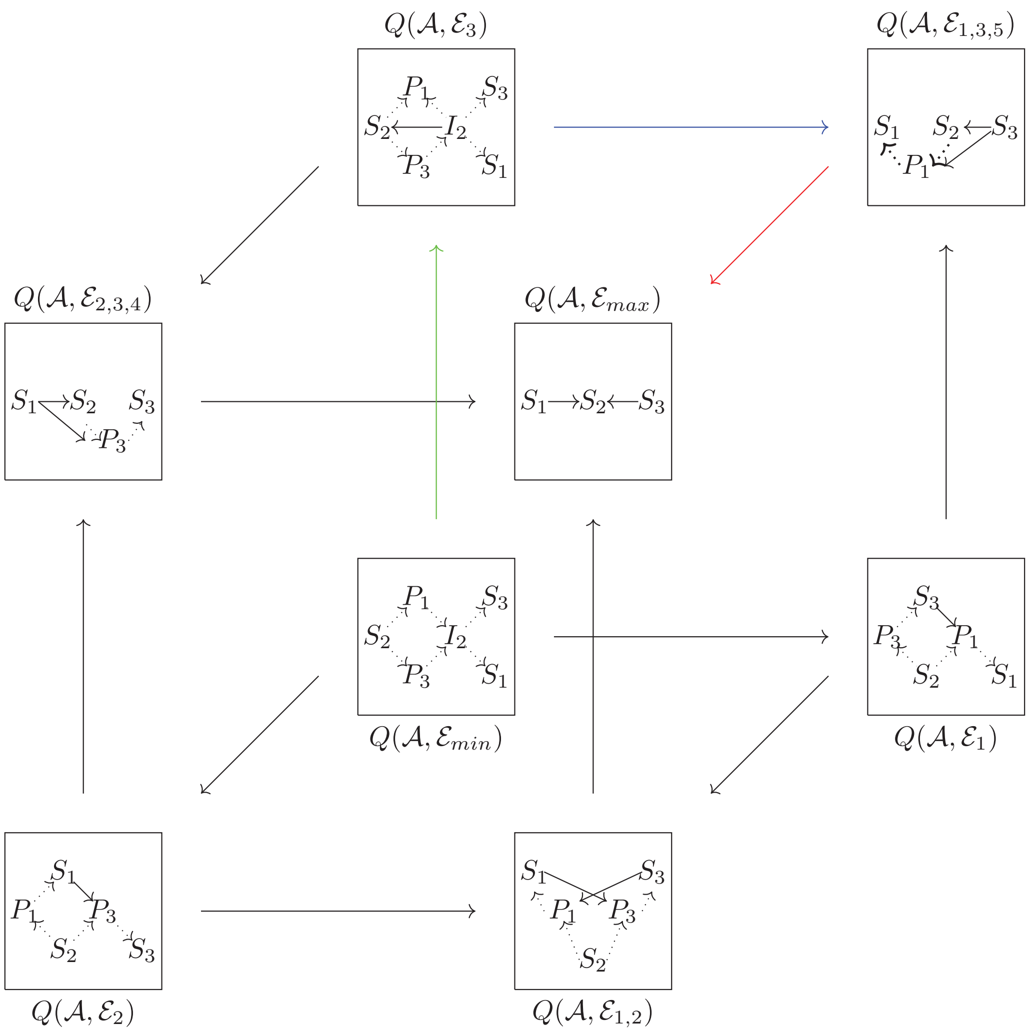
We have exactly $2^3 = 8$ exact structures since $\mathcal{A}$ admits 3 Auslander-Reiten sequences, and every exact structure is uniquely determined by a choice of a set of Auslander-Reiten sequences, see 5.7.

Hence the lattice of exact structures $Ex(\mathcal{A})$ is a cube, where the oriented arrows present inclusions:



Furthermore, the following diagram describes the quiver of the exact category 3.8, associated to each exact structure in the previous example.

Compare also the biquivers in [17], example 4.56.



We can see that the path of matrix reductions discussed in section 4.1 corresponds to the chain of exact structures

$$\mathcal{E}_{max} \supset \mathcal{E}_{1,3,5} \supset \mathcal{E}_3 \supset \mathcal{E}_{min}.$$

In fact, the ad hoc notion of a quiver of a matrix problem, given by the algebra operating on the current reduced form, together with arrows of degree one corresponding to the unreduced blocks, can finally be made precise: The reduction of exact structures transforms the Gabriel quiver $Q(\mathcal{A}, \mathcal{E}_{max})$ into the Auslander-Reiten quiver $Q(\mathcal{A}, \mathcal{E}_{min})$, and, in the first reduction step, the quiver of the exact category $(\mathcal{A}, \mathcal{E}_{1,3,5})$ coincides with the quiver depicted after the first reduction step in 4.1. We only need to make precise why the exact category $(\mathcal{A}, \mathcal{E}_{1,3,5})$ corresponds to reducing the block A of the matrix problem $M = [A|B]$. This is done in the next section 4.3.

4.3. Constructing new exact structures from given ones

One method to produce exact structures is using exact functors, see also [9, section 1.4]:

Definition 4.3. Let $(\mathcal{A}, \mathcal{E}_\mathcal{A})$ and $(\mathcal{B}, \mathcal{E}_\mathcal{B})$ be exact categories and let $F : (\mathcal{A}, \mathcal{E}_\mathcal{A}) \to (\mathcal{B}, \mathcal{E}_\mathcal{B})$ be an exact functor, that is, the image (Fi, Fd) of each exact pair (i, d) in $(\mathcal{A}, \mathcal{E}_\mathcal{A})$ is exact in $(\mathcal{B}, \mathcal{E}_\mathcal{B})$. We define the following subclass of $\mathcal{E}_\mathcal{A}$:

$$\mathcal{E}_F = \{\xi \in \mathcal{E}_\mathcal{A} \ : \ 0 \to A \to B \to C \to 0 \mid F(\xi) \text{ is split exact in } \mathcal{B}\}$$

The following is a reformulation in our context of [9, Lemma 1.9], and it also follows from [13, 7.3], see [6, Exercise 5.3].

Proposition 4.4. *Let $F : (\mathcal{A}, \mathcal{E}_\mathcal{A}) \to (\mathcal{B}, \mathcal{E}_\mathcal{B})$ be an exact functor. Then $(\mathcal{A}, \mathcal{E}_F)$ is an exact category.*

Proof. We verify that $\mathcal{E}_F$ satisfies the axioms of an exact structure on $\mathcal{A}$. Since $F(1_A) = 1_{FA}$ for every object A of $\mathcal{A}$ and the identity is admissible monic and epic, $\mathcal{E}_F$ satisfies (E0) and (E0)^{op}.

An admissible monic in $\mathcal{E}_F$ is a morphism i in a pair (i, d) in $\mathcal{E}_\mathcal{A}$ such that $F(i)$ is an admissible monic for the split exact structure $\mathcal{E}_{min}(\mathcal{B})$ on $\mathcal{B}$. Since $\mathcal{E}_{min}(\mathcal{B})$ is closed under composition of admissible monics we conclude that $\mathcal{E}_F$ satisfies (E1). The dual argument applies to admissible epics.

Now let us verify that $\mathcal{E}_F$ satisfies (E2) and (E2)^{op}: The push-out of an admissible monic $i : A \rightarrowtail B$ in $\mathcal{E}_F$ along an arbitrary morphism $f : A \to X$ exists in the exact category $(\mathcal{A}, \mathcal{E}_\mathcal{A})$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, "g"] \\ X \arrow[r, "j"] & D \end{tikzcd}$$

We need to verify that j is an admissible monic not only for $\mathcal{E}_\mathcal{A}$, but also for $\mathcal{E}_F$. Consider the commutative diagram in $(\mathcal{A}, \mathcal{E}_\mathcal{A})$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "d"] \arrow[d, "g"] & C \arrow[d, equal] & & \xi \\ X \arrow[r, "j"] & D \arrow[r, "d'"] & C & & \xi' = f \cdot \xi \end{tikzcd}$$

which is mapped under the functor F to the commutative diagram in $\mathcal{B}$

$$\begin{tikzcd} F(A) \arrow[r, "F(i)"] \arrow[d, "F(f)"'] & F(B) \arrow[r, "F(d)"] \arrow[d, "F(g)"] & F(C) \arrow[d, equal] & F(\xi) \in \mathcal{E}_{min}(\mathcal{B}) \\ F(X) \arrow[r, "F(j)"] & F(D) \arrow[r, "F(d')"] & F(C) & F(\xi') \end{tikzcd}$$

Since i is an admissible monic, we know that $F(i)$ is a section and $F(d)$ a retraction. By commutativity, $F(d')F(g) = F(d)$ is a retraction, so $F(d')$ is a retraction. Since F is exact, the pair $(F(j), F(d'))$ is exact, hence $F(j)$ is a section and (E2) holds.

The dual argument applied to the pull-back diagram of an admissible epic yields (E2)^{op}. $\square$

The basic idea of matrix reduction is to fix a subproblem and completely reduce representations of this subproblem into direct sums of indecomposables. On the level of exact structures, having nothing but direct sums of indecomposables corresponds to the choice of the split exact structure. Thus, if the functor F in Definition 4.3 is the projection onto a suitable subcategory (like representations of a subquiver or modules over a subalgebra), the definition of the exact structure $\mathcal{E}_F$ corresponds to the idea that objects in the subcategory are completely reduced into sums of indecomposables (we consider those exact sequences ξ whose projection $F(\xi)$ is split exact).

In [9, section 4], several classical reductions are discussed, like one-point extension of an algebra or reduction of modules to a vector space problem. The underlying procedure is always the same: complete reduction on a subproblem corresponds to the choice of an exact structure on the original problem, composed of those short exact sequences that split when restricted to the subproblem. We refer to [9] for more details. However the examples discussed there consider only *one* choice of exact structure on the original category $\operatorname{mod} A$. We propose to iterate this process (as it is done for matrix reductions or for Roiter's bocses) that is, to consider a chain of exact structures on the same underlying category $\mathcal{A}$.

We return now to the Example 4.2, the category $\mathcal{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

Consider $\mathcal{A}$ equipped with the abelian exact structure $\mathcal{E}_{max}$, and set up the first reduction step: Let $\mathcal{B} = \operatorname{rep} Q'$ be the category of representations of the subquiver

$$Q': \qquad 1 \xrightarrow{\alpha} 2$$

of Q, and let

$$F : \operatorname{rep} Q \to \operatorname{rep} Q'$$

be the restriction functor. Thus the exact structure $\mathcal{E}_F$ on $\mathcal{A}$ is given by all short exact sequences $0 \to U \to V \to W \to 0$ in $\operatorname{rep} Q$ whose restriction to the subquiver Q' is split. It is not difficult to verify that exactly the non-split short exact sequences numerated 1, 3 and 5 from the table in Example 4.2 are those whose restriction to Q' splits. We therefore conclude that

$$\mathcal{E}_F = \mathcal{E}_{1,3,5}.$$

The matrix reduction step (1) in section 4.1 was to reduce the matrix A. In view of the theory developed by now, this corresponds to choosing the exact structure which splits on the subquiver supported by the arrow α, that is, the quiver Q'. Therefore, the reduction step (1) corresponds precisely to the reduction of exact structures

$$\mathcal{E}_{max} \to \mathcal{E}_{1,3,5}$$

on $\mathcal{A} = \operatorname{rep} Q$.

5. The lattice of exact structures of an additive category

Definition 5.1. Let $\mathcal{A}$ be an additive category. We denote by $(Ex(\mathcal{A}), \subseteq)$ the poset of exact structures $\mathcal{E}$ on $\mathcal{A}$, where the partial order is given by containment $\mathcal{E}' \subseteq \mathcal{E}$.

This *containment* partial order is the *reduction of exact structures* discussed in 4.1.

Lemma 5.2. *For a family of exact structures $(\mathcal{E}_\omega)_{\omega\in\Omega}$ on an additive category $\mathcal{A}$, the intersection*

$$\bigcap_{\omega\in\Omega} \mathcal{E}_\omega = \{\xi | \xi \in \mathcal{E}_\omega \textit{ for all } \omega \in \Omega\}$$

forms an exact structure on $\mathcal{A}$.

Proof. Let us show that this class verifies the axioms of the Definition 2.1: (E0), (E0)^{op}, (E1) and (E1)^{op} are satisfied since every $\mathcal{E}_\omega$ satisfies these axioms. For (E2), the push-out of an admissible monic i in $\mathcal{E}_\omega$ exists in $\mathcal{E}_\omega$ and yields an admissible monic f_i in $\mathcal{E}_\omega$ for all $\omega \in \Omega$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "g"] \\ C \arrow[r, tail, "f_i"] & D \end{tikzcd}$$

Since the push-out is unique up to isomorphism, and an exact structure is closed under isomorphisms, we conclude that (E2) satisfied. Dually for (E2)^{op}. $\square$

Theorem 5.3. *The poset of exact structures of an additive category $\mathcal{A}$ is a lattice $(Ex(\mathcal{A}), \subseteq, \bigwedge, \bigvee)$.*

Proof. Using Lemma 5.2 we define the following two binary operations on the poset $Ex(\mathcal{A})$; *the meet* $\bigwedge$ is defined by $\mathcal{E}_\omega \bigwedge \mathcal{E}_{\omega'} = \mathcal{E}_\omega \cap \mathcal{E}_{\omega'}$, and *the join* $\bigvee$ is defined by

$$\mathcal{E}_\omega \bigvee \mathcal{E}_{\omega'} = \bigcap \{\mathcal{E} \in Ex(\mathcal{A}) \mid \mathcal{E}_\omega \subseteq \mathcal{E}, \mathcal{E}_{\omega'} \subseteq \mathcal{E}\}.$$

Note that the intersection defining the join is not an empty set since it contains the maximal exact structure $\mathcal{E}_{max}$ of the additive category $\mathcal{A}$; see 2.14 for the existence of $\mathcal{E}_{max}$. Conclude that the poset $Ex(\mathcal{A})$ is a lattice since it is a $\bigwedge$-semilattice and a $\bigvee$-semilattice. $\square$

Corollary 5.4. *The lattice $(Ex(\mathcal{A}), \subseteq, \bigwedge, \bigvee)$ of exact structures of an additive category is bounded and complete.*

Proof. The lattice is bounded since it has a *top* $\mathcal{E}_{max}$ and a *bottom* $\mathcal{E}_{min}$ verifying

$$\mathcal{E} \bigwedge \mathcal{E}_{max} = \mathcal{E} \text{ and } \mathcal{E} \bigvee \mathcal{E}_{min} = \mathcal{E}$$

for any exact structure $\mathcal{E}$ in $Ex(\mathcal{A})$. And it is complete since all subsets $\{(\mathcal{E}_\omega)_{\omega\in\Omega}\}$ of $Ex(\mathcal{A})$ have both a *meet* $\bigwedge(\mathcal{E}_\omega)_{\omega\in\Omega} = \cap(\mathcal{E}_\omega)_{\omega\in\Omega}$ and a *join* defined by $\bigvee(\mathcal{E}_\omega)_{\omega\in\Omega} = \bigcap\{\mathcal{E} | \mathcal{E}_\omega \subseteq \mathcal{E}, \forall\omega \in \Omega\}$, by Lemma 5.2. $\square$

Example 5.5. As seen in Example 2.19, if $\mathcal{A}$ is a triangulated category, then the lattice of exact structures is a single point: $Ex(\mathcal{A}) = \{\mathcal{E}_{min}\}$.

Example 5.6. Consider the category $\mathcal{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

then the lattice of exact structures $Ex(\mathcal{A})$ is the *cube* we construct in the Example 4.2. Let us mention that by taking other forms of the quiver of type A_3 such as

$$Q: \qquad 1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3$$

or

$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

we get a similar *cube* (that is, a Boolean lattice) for $Ex(\mathcal{A})$.

In general, if we have n Auslander-Reiten sequences then we have exactly 2^n exact structures which is the power set cardinality of an n elements set. In fact, the lattice is Boolean for a large class of exact categories:

Theorem 5.7. *[10] Let $\mathcal{A}$ be a skeletally small, Hom-finite, idempotent complete additive category which has finitely many indecomposable objects up to isomorphism. Then the lattice of exact structures $Ex(\mathcal{A})$ is Boolean.*

In fact, the set of exact structures on $\mathcal{A}$ is in bijection with the power set of Auslander-Reiten sequences in $\mathcal{A}$.

Proof. This follows directly from 2.7 (see also 3.1, 3.7 and 3.10) in the work of Enomoto [10]. □

6. Length function on the poset *Obj* $\mathcal{A}$

The aim of this section is to define and study the notion of length for objects of an exact category $(\mathcal{A}, \mathcal{E})$. Contrary to abelian categories, the Jordan-Hölder property does not hold for general exact (additive) categories, which makes it impossible to define length using composition series.

Throughout this section, we assume that $\mathcal{A}$ is essentially small, and we denote by $Obj\,\mathcal{A}$ the set of isomorphism classes of objects in $\mathcal{A}$. We show that the notion of $\mathcal{E}$-subobjects allows to turn $Obj\,\mathcal{A}$ into a poset, and that the length of an object corresponds to the height function of this poset. Since the exact structure $\mathcal{E}$ is closed under isomorphisms, we work mostly with objects X rather than their isomorphism classes $[X] \in Obj\,\mathcal{A}$.

6.1. The length function

Definition 6.1. We define the $\mathcal{E}$-length function $l_{\mathcal{E}} : Obj\,\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ as supremum over the lengths of chains of admissible monics which are not isomorphisms.

That is, for an object X of $(\mathcal{A}, \mathcal{E})$, one has $l_{\mathcal{E}}(X) = n \in \mathbb{N}$ if n is the maximal length of a chain of admissible monics which are not isomorphisms

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_{n-1} \rightarrowtail X_n = X.$$

We say in this case that X has finite $\mathcal{E}$-length, or that X is $\mathcal{E}$-finite. If no such bound exists, we say that X has infinite length, or $l_{\mathcal{E}}(X) = \infty$. Clearly, isomorphic objects have the same length, and therefore this definition gives rise to a function $l_{\mathcal{E}} : Obj\,\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ defined on isomorphism classes.

Remark 6.2. The $\mathcal{E}$-simple objects are precisely those of length $l_{\mathcal{E}}(X) = 1$.

Example 6.3. We illustrate how the $\mathcal{E}$-length of an object changes with the exact structure by considering the indecomposable injective representation I_2 from the example discussed in 4.2, and measure its length with respect to various exact structures from $Ex(\mathcal{A})$, see 4.2:

$$l_{\mathcal{E}_{min}}(I_2) = 1$$
$$l_{\mathcal{E}_{1,3,5}}(I_2) = 2$$
$$l_{\mathcal{E}_{ab}}(I_2) = 3.$$

We call an exact category $(\mathcal{A}, \mathcal{E})$ *finite* if every object is $\mathcal{E}$-finite. This is equivalent to the condition that $\mathcal{A}$ is an $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian category in the following sense:

Definition 6.4. An object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Noetherian if any increasing sequence of $\mathcal{E}$-subobjects of X

$$X_1 \rightarrowtail X_2 \rightarrowtail \cdots \rightarrowtail X_{n-1} \rightarrowtail X_n \rightarrowtail X_n \cdots$$

becomes stationary. Dually, an object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artinian if any descending sequence of $\mathcal{E}$-subobjects of X

$$\cdots X_n \rightarrowtail X_n \rightarrowtail X_{n-1} \rightarrowtail \cdots \rightarrowtail X_2 \rightarrowtail X_1$$

becomes stationary. The exact category $(\mathcal{A}, \mathcal{E})$ is called $\mathcal{E}$-Artinian (respectively $\mathcal{E}$-Noetherian) if every object is $\mathcal{E}$-Artinian (respectively $\mathcal{E}$-Noetherian).

Proposition 6.5 *($\mathcal{E}$-Hopkins–Levitzki theorem). An object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian if and only if it is $\mathcal{E}$-finite.*

Proof. For an $\mathcal{E}$-finite object X of length $l_{\mathcal{E}}(X) = n \in \mathbb{N}$, the longest chain of proper admissible monics is of length n. Thus any increasing or decreasing sequence of $\mathcal{E}$-subobjects of X must become stationary and X is $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian.

Conversely, let X be an $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian object. Then any increasing chain of proper admissible monics ending with X has to be of finite length. So X is $\mathcal{E}$-finite. □

We now study how the length function behaves with respect to short exact sequences: It turns out to be a superadditive function. We provide in 6.9 an example that it need not be additive in general.

Theorem 6.6. *Let $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ be a short exact sequence of $\mathcal{E}$-finite objects. Then*

$$l_{\mathcal{E}}(Y) \geq l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Z).$$

Proof. Consider a chain of proper admissible monics which defines the length s of Z:

$$0 = Z_0 \overset{i_1}{\rightarrowtail} Z_1 \rightarrowtail \cdots \rightarrowtail Z_{s-1} \overset{i_s}{\rightarrowtail} Z_s = Z.$$

Denote by Y_{s-1} the pull-back of g along i_s. By the dual of [6, Prop 2.12], there exists a commutative diagram with exact columns

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, tail, "f_{s-1}"'] & X \arrow[d, tail, "f"] \\
Y_{s-1} \arrow[r, tail, "j_s"] \arrow[d, "g_{s-1}"'] & Y \arrow[d, "g"] \\
Z_{s-1} \arrow[r, tail, "i_s"'] & Z
\end{tikzcd}$$

Since i_s is an admissible monic, [6, Prop 2.15] yields that j_s is one as well, and since i_s is not an isomorphism, j_s cannot be an isomorphism by [6, 3.3]. Iterated pull-backs along the morphisms $g_{s-1}, g_{s-2}, \dots, g_1$ therefore yield the following exact diagram with exact columns and proper admissible monics $j_1, \dots j_s$:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, tail, "f_0"'] & X \arrow[r, dotted, no head] \arrow[d, tail, "f_1"'] & X \arrow[r, equal] \arrow[d, tail, "f_{s-1}"'] & X \arrow[d, tail, "f"] \\
Y_0 \arrow[r, tail, "j_1"] \arrow[d, "g_0"'] & Y_1 \arrow[r, dotted, no head] \arrow[d, "g_1"'] & Y_{s-1} \arrow[r, tail, "j_s"] \arrow[d, "g_{s-1}"'] & Y \arrow[d, "g"] \\
0 = Z_0 \arrow[r, tail, "i_1"'] & Z_1 \arrow[r, dotted, no head] & Z_{s-1} \arrow[r, tail, "i_s"'] & Z
\end{tikzcd}$$

Note that f_0 is admissible monic with zero cokernel, hence an isomorphism by Remark 3.2. Composing the sequence of proper admissible monics $j_1, \dots j_s$ with any sequence of proper subobjects of X:

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_{r-1} \rightarrowtail X_r = X$$

yields a chain of proper admissible monics ending in Y of length $r + s$. Hence, the definition of length yields $l_{\mathcal{E}}(Y) \geq l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Z)$. $\square$

Corollary 6.7. *Let $Y \xrightarrow{g} Z$ be an admissible epic between $\mathcal{E}$-finite objects. Then $l_{\mathcal{E}}(Y) \geq l_{\mathcal{E}}(Z)$.*

Proof. The kernel of g yields the short exact sequence $\ker g \rightarrowtail Y \xrightarrow{g} Z$. Hence Theorem 6.6 implies that $l_{\mathcal{E}}(Y) \geq l_{\mathcal{E}}(Z)$ since $l_{\mathcal{E}}(\ker g) \geq 0$. $\square$

Remark 6.8. Analogously to abelian categories, one could define a *composition series* of an object X to be a chain of admissible monics

$$0 = X_0 \overset{i_1}{\rightarrowtail} X_1 \overset{i_2}{\rightarrowtail} \cdots \overset{i_{n-1}}{\rightarrowtail} X_{n-1} \overset{i_n}{\rightarrowtail} X_n = X$$

whose cokernels are $\mathcal{E}$-simple. These composition series are certainly chains of proper admissible monics that cannot be refined, so they are good candidates for chains defining the length of X. However, the length of a composition series of an object X need not be unique in general, that is, the Jordan-Hölder property does not hold necessarily. We provide a simple example:

Example 6.9. Consider the split exact structure $\mathcal{E} = \mathcal{E}_{min}$. As seen in Remark 3.4, the $\mathcal{E}$-simple objects are precisely the indecomposables. Hence in this case the $\mathcal{E}$-length function measures the maximum number of indecomposable direct summands of an object X. The Jordan-Hölder property thus coincides with the Krull-Schmidt property, and we obtain a counterexample re-visiting Example 3.5: The category $\mathcal{A}_S$ for $S = \mathbb{N}\backslash\{1\}$ equipped with the split exact structure admits two $\mathcal{E}$-simple objects, k^2 and k^3, up to isomorphism. There are two composition series for the object $X = k^6$ in $\mathcal{A}_S$, one of length 3 with cokernels k^2, the other of length 2 with cokernels k^3. Following our definition, the object $X = k^6$ has length $l_{\mathcal{E}}(X) = 3$.

This example also shows that the length function need not be additive on short exact sequences: Consider the short exact sequence

$$0 \to k^3 \rightarrowtail k^6 \twoheadrightarrow k^3 \to 0$$

in $(\mathcal{A}_S, \mathcal{E})$, then

$$l_{\mathcal{E}}(k^6) = 3 \neq 2 = l_{\mathcal{E}}(k^3) + l_{\mathcal{E}}(k^3).$$

6.2. The poset structure on Obj $\mathcal{A}$

We assume in this section that $(\mathcal{A}, \mathcal{E})$ is a finite exact category, that is, every object is $\mathcal{E}$-finite. In general the length function behaves well with respect to subobjects, that is $l_{\mathcal{E}}(X) \leq l_{\mathcal{E}}(Y)$ if $X \subset_{\mathcal{E}} Y$. The following lemma shows that strict inclusion is also preserved when the objects are of *finite* length:

Lemma 6.10. *Consider two objects X and Y in A such that $X \subsetneq_{\mathcal{E}} Y$. Then*

$$l_{\mathcal{E}}(X) \lneqq l_{\mathcal{E}}(Y).$$

Proof. Let X be a proper admissible subobject of Y, that is, there exists an admissible monic $X \overset{i}{\rightarrowtail} Y$ which is not an isomorphism. We show that

$$l_{\mathcal{E}}(X) + 1 \leq l_{\mathcal{E}}(Y).$$

Assume that X has length $l_{\mathcal{E}}(X) = n$. Extending a chain of subobjects defining $l_{\mathcal{E}}(X)$, we obtain a sequence of proper admissible monics ending via i in Y of the following form:

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_{n-1} \rightarrowtail X = X_n \overset{i}{\rightarrowtail} Y.$$

Thus the length of Y is at least $n + 1$. □

The previous lemma allows us to show that the notion of $\mathcal{E}$-subobjects turns *Obj* $\mathcal{A}$ into a poset:

Proposition 6.11. *Let $(\mathcal{A}, \mathcal{E})$ be a finite essentially small exact category. Then the relation $\subset_{\mathcal{E}}$ induces a partial order on Obj $\mathcal{A}$.*

Proof. We defined the relation $\subset_{\mathcal{E}}$ on objects, but since the exact structure $\mathcal{E}$ is closed under isomorphisms, one also obtains a well-defined relation on the set of isomorphism classes *Obj* $\mathcal{A}$. It remains to show that this relation verifies the three properties defining a partial order. We do so mostly by working with objects X rather than their isomorphism classes $[X]$.

1. Reflexive: $X \subset_{\mathcal{E}} X$ since the identity $X \overset{id_X}{\rightarrowtail} X$ is an admissible monic by (E0).

2. Antisymmetric: Assume that $X \subset_{\mathcal{E}} Y$ and $Y \subset_{\mathcal{E}} X$. Then we have $l_{\mathcal{E}}(X) \le l_{\mathcal{E}}(Y)$ and $l_{\mathcal{E}}(Y) \le l_{\mathcal{E}}(X)$, and so $l_{\mathcal{E}}(X) = l_{\mathcal{E}}(Y)$. Hence the admissible monic $X \rightarrowtail Y$ establishing $X \subset_{\mathcal{E}} Y$ cannot be proper by Lemma 6.10, which shows $X = Y$ in $Obj\,\mathcal{A}$.
3. Transitive: if $X \subset_{\mathcal{E}} Y$ and $Y \subset_{\mathcal{E}} Z$ then there exist admissible monics $X \overset{f}{\rightarrowtail} Y$ and $Y \overset{f'}{\rightarrowtail} Z$. By (E1), $X \overset{f' \circ f}{\rightarrowtail} Z$ is an admissible monic and so $X \subset_{\mathcal{E}} Z$. $\square$

Remark 6.12. Now since we know that the notion of $\mathcal{E}$-subobject induces a poset structure on $Obj\,\mathcal{A}$, we could define the length of an object X as the height of the element $[X]$ in $Obj\,\mathcal{A}$. In fact, this is exactly how we defined length (as maximum length of a chain of $\mathcal{E}$-subobjects), except that we rather start with the zero object having length zero, instead of height one.

We recall the following definition from [16]:

Definition 6.13. A *measure for a poset* $\mathcal{S}$ is a morphism of posets $\mu : \mathcal{S} \to \mathcal{P}$ where $(\mathcal{P}, \le)$ is a totally ordered set.

Theorem 6.14. *The length function $l_{\mathcal{E}}$ of a finite essentially small exact category $(\mathcal{A}, \mathcal{E})$ is a measure for the poset $Obj\,\mathcal{A}$.*

Proof. The length function $l_{\mathcal{E}} : Obj\,\mathcal{A} \to \mathbb{N}$ is defined on the set $Obj\,\mathcal{A}$, which is a partially ordered set by Proposition 6.11. Moreover, $l_{\mathcal{E}}$ is a morphism of partially ordered sets, and so $l_{\mathcal{E}}$ is a measure since $(\mathbb{N}, \le)$ is totally ordered. $\square$

7. Gabriel-Roiter measure

In his proof of the first Brauer-Thrall conjecture [23], Roiter used an induction scheme which Gabriel formalized in his report on abelian length categories [11]. This so-called Gabriel-Roiter measure on module categories was further studied by Ringel in [21] and [22] in the representation-infinite case. Later Krause presented an axiomatic characterization of the Gabriel-Roiter measure on abelian length categories which reveals its combinatorial nature in [15] and [16]. Our aim in this section is to extend the work of [16] to the more general context of exact categories. Most of the results presented here generalize the corresponding version of Ringel or Krause.

In this section we consider $(\mathcal{A}, \mathcal{E}, ind\,\mathcal{A}, l_{\mathcal{E}})$ where $\mathcal{A}$ is an essentially small additive category, $\mathcal{E}$ is a fixed exact structure such that $(\mathcal{A}, \mathcal{E})$ is a finite exact category, $ind\,\mathcal{A}$ is the set of isomorphism classes of indecomposable objects of $\mathcal{A}$, and $l_{\mathcal{E}}$ is the associated length. The set $ind\,\mathcal{A}$ does not depend on the exact structure $\mathcal{E}$, but the partial order does depend on $\mathcal{E}$. We therefore write $(ind\,\mathcal{A}, \subset_{\mathcal{E}})$ when referring to the poset.

7.1. The definition and existence

The following definition extends the one from [16, Definition 1.6] to the realm of exact categories: a Gabriel-Roiter measure on $(ind\,\mathcal{A}, \subset_{\mathcal{E}})$ is a morphism of partially ordered sets which refines the length function $l_{\mathcal{E}}$ and satisfies that the measure of an object X cannot exceed the measure of an object Y of at most equal length if all subobjects of X have smaller measure than Y:

Definition 7.1. A map $\mu_{\mathcal{E}} : (ind\,\mathcal{A}, \subset_{\mathcal{E}}) \to (\mathcal{P}, \le)$ is called a Gabriel-Roiter measure on the exact category $(\mathcal{A}, \mathcal{E})$ if it verifies the following axioms

(GR_1) $\mu_{\mathcal{E}}$ is a measure
(GR_2) $\mu_{\mathcal{E}}(X) = \mu_{\mathcal{E}}(Y)$ implies $l_{\mathcal{E}}(X) = l_{\mathcal{E}}(Y)$ for all $X, Y \in ind\,\mathcal{A}$
(GR_3) If $l_{\mathcal{E}}(X) \geq l_{\mathcal{E}}(Y)$ and $\mu_{\mathcal{E}}(X') \lneqq \mu_{\mathcal{E}}(Y)$ for all $X' \subsetneq_{\mathcal{E}} X$, then

$$\mu_{\mathcal{E}}(X) \leq \mu_{\mathcal{E}}(Y).$$

Most constructions of a Gabriel-Roiter measure use as totally ordered set $(\mathcal{P}, \leq)$ the set $\mathfrak{S}(\mathbb{N})$ of all vectors of natural numbers of finite length equipped with the lexicographic order $\lll$ on vectors with the natural order on $\mathbb{N}$ reversed. More explicitly, let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$ be two vectors of natural numbers. We write $x \lll\!\!\!\!\not\;\; y$ if the element x in the ordered set $\mathfrak{S}(\mathbb{N})$ is smaller but not equal to y. To compare these two vectors by $\lll$, we begin with the first elements; if for example $x_1 = y_1$ we pass to the second elements, if again $x_2 = y_2$ we pass to the third, and we continue like this until we obtain one of the following three cases:

1. if $x_k = y_k$ for all $1 \leq k \leq i-1$ and at position i there are two different elements $x_i \lneqq y_i$ in $(\mathbb{N}, \leq)$, then we get the inverse relation for the vectors: $(x_1, ..., x_n) \not\!\!\ggg (y_1, ..., y_m)$
2. if $n \lneqq m$ and $x_k = y_k$ for all $1 \leq k \leq n$, then $(x_1, ..., x_n) \lll\!\!\!\!\not\;\; (y_1, ..., y_m)$
3. if $m = n$ and $x_k = y_k$ for all $1 \leq k \leq m$, then $(x_1, ..., x_n) = (y_1, ..., y_m)$.

More loosely speaking, one has $x \lll y$ if x is a subword of y in the sense of point 2 above, or y is denser than x at the beginning, for example

$$(1) \lll (1, 3, 4) \lll (1, 2, 4).$$

Let us now consider the following construction (we show later that it yields a Gabriel-Roiter measure for exact categories). For a fixed indecomposable object $X \in \mathcal{A}$, we consider the proper $\mathcal{E}$-filtrations $F_{\mathcal{E}}(X)$ of X

$$F_{\mathcal{E}}(X) = X_1 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_n = X$$

where all objects X_i are indecomposable. Denote the vector of lengths in this filtration by

$$l_{\mathcal{E}}(F_{\mathcal{E}}(X)) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X)).$$

Definition 7.2. Define a map

$$\mu_{\mathcal{E}} : (ind\,\mathcal{A}, \subset_{\mathcal{E}}) \to (\mathfrak{S}(\mathbb{N}), \lll)$$

by

$$X \longmapsto \mu_{\mathcal{E}}(X) = \max_{F_{\mathcal{E}}(X)} (l_{\mathcal{E}}(F_{\mathcal{E}}(X)))$$

where the maximum is over all proper $\mathcal{E}$-filtrations of X by indecomposables. Note that the maximum is attained: We know by Lemma 6.10 that $l_{\mathcal{E}}(F_{\mathcal{E}}(X))$ is a strictly increasing sequence. But there are only finitely many strictly increasing sequences ending in the natural number $l_{\mathcal{E}}(X)$.

Example 7.3. Consider the split exact structure $\mathcal{E}_{min}$. Then all the indecomposable objects are $\mathcal{E}_{min}$-simples, and $l_{\mathcal{E}_{min}}(X) = 1$, therefore

$$\mu_{\mathcal{E}_{min}}(X) = (1)$$

for all $X \in ind\,\mathcal{A}$.

This is the case for Example 3.5; $X = K^2$ or $X = K^3$ and then

$$\mu_{\mathcal{E}_{min}}(K^2) = \mu_{\mathcal{E}_{min}}(K^3) = (1).$$

The following lemma can be derived from [16, section 1], applied to the length function $l_{\mathcal{E}}$ on the poset $(ind\mathcal{A}, \subset_{\mathcal{E}})$. We give a short proof in our setup.

Lemma 7.4. $\mu_{\mathcal{E}} : (ind\,\mathcal{A}, \subset_{\mathcal{E}}) \to (\mathfrak{S}(\mathbb{N}), \lll)$ *is a measure for* $(ind\,\mathcal{A}, \subset_{\mathcal{E}})$.

Proof. We have that $(ind\,\mathcal{A}, \subset_{\mathcal{E}})$ is a partially ordered set by the induced order on $ind\,\mathcal{A} \subset Obj\,\mathcal{A}$, and it is easy to see that $(\mathfrak{S}(\mathbb{N}), \lll)$ is a totally ordered set since $(\mathbb{N}, \leq)$ is totally ordered. It suffices to show that $\mu_{\mathcal{E}}$ is a morphism of posets. To this end, let $X \subsetneq_{\mathcal{E}} Y$, and consider a filtration

$$F_{\mathcal{E}}(X) : X_1 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_n = X$$

such that

$$\mu_{\mathcal{E}}(X) = l_{\mathcal{E}}(F_{\mathcal{E}}(X)) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_n)).$$

This yields the following filtration of Y

$$F_{\mathcal{E}}(Y) : X_1 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_n = X \subsetneq_{\mathcal{E}} Y$$

with

$$l_{\mathcal{E}}(F_{\mathcal{E}}(Y)) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_n), l_{\mathcal{E}}(Y))$$

hence

$$\mu_{\mathcal{E}}(X) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_n)) \lll (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_n), l_{\mathcal{E}}(Y)) \lll \mu_{\mathcal{E}}(Y). \quad \square$$

The previous lemma establishes that the measure of a subobject X' of X is smaller than the measure of X. Of particular importance will be subobjects of X whose measure is a subword of the measure of X, we call them as follows:

Definition 7.5. A chain

$$F_{\mathcal{E}}(X) : \; X_1 \subsetneq_{\mathcal{E}} X_2 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_{n-1} \subsetneq_{\mathcal{E}} X_n = X$$

in $ind\,\mathcal{A}$ is called a $\mu_{\mathcal{E}}$*-filtration* of X if for all $1 \leq i \leq n$ the vector $\mu_{\mathcal{E}}(X_i)$ coincides with the subword of $\mu_{\mathcal{E}}(X)$ formed by the first i entries.

Lemma 7.6. *Let* $F_{\mathcal{E}}(X) : \; X_1 \subsetneq_{\mathcal{E}} X_2 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_{n-1} \subsetneq_{\mathcal{E}} X_n = X$ *be a filtration of* X *realizing the measure of* X*, that is,* $\mu_{\mathcal{E}}(X) = l_{\mathcal{E}}(F_{\mathcal{E}}(X))$*. Then* $F_{\mathcal{E}}(X)$ *is a* $\mu_{\mathcal{E}}$*-filtration of* X*.*

Proof. We have to show for each $1 \leq i \leq n$ that $\mu_{\mathcal{E}}(X_i) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_i))$. Of course, the sequence $(l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_i))$ is one candidate for the maximum $\mu_{\mathcal{E}}(X_i)$, so we only need to show that the case

$$\mu_{\mathcal{E}}(X_i) = (l_{\mathcal{E}}(X'_1), ..., l_{\mathcal{E}}(X'_m)) \ggg\!\!\!\!\!\!\not\;\; (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_i)) \text{ with } X'_m = X_i$$

is impossible. By definition of the order relation $\lll$, there are two situations to be considered:

1. there exists an index $1 \leq j \leq \min\{i, m\}$ such that

$$l_{\mathcal{E}}(X_k) = l_{\mathcal{E}}(X'_k) \text{ for all } 1 \leq k < j \text{ and } l_{\mathcal{E}}(X'_j) < l_{\mathcal{E}}(X_j).$$

   But then the filtration of X

$$X'_1 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X'_m = X_i \subsetneq_{\mathcal{E}} X_{i+1} \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_n = X$$

   yields a length sequence which is denser in the beginning than $\mu_{\mathcal{E}}(X)$, which contradicts the fact that $F_{\mathcal{E}}(X)$ realizes the measure of X.
2. The sequence $(l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_i))$ is a subword of $(l_{\mathcal{E}}(X'_1), ..., l_{\mathcal{E}}(X'_m))$, that is, $i \lneqq m$ and $l_{\mathcal{E}}(X_k) = l_{\mathcal{E}}(X'_k)$ for all $1 \leq k \leq i$. But then again the same filtration of X in 1 yields a contradiction. □

Theorem 7.7. *(Compare [16, Section 3.1]). The map $\mu_{\mathcal{E}}$ is a Gabriel-Roiter measure for ind$(\mathcal{A}, \mathcal{E})$.*

Proof. We verify that $\mu_{\mathcal{E}}$ as given in Definition 7.2 satisfies the three axioms of a Gabriel-Roiter measure.

(GR_1): This is Lemma 7.4.

(GR_2): If $\mu_{\mathcal{E}}(X) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_n)) = (l_{\mathcal{E}}(Y_1), ..., l_{\mathcal{E}}(Y_m)) = \mu_{\mathcal{E}}(Y)$ then clearly $l_{\mathcal{E}}(X) = l_{\mathcal{E}}(X_n) = l_{\mathcal{E}}(Y_m) = l_{\mathcal{E}}(Y)$.

(GR_3): Let X and Y be such that $l_{\mathcal{E}}(X) \geq l_{\mathcal{E}}(Y)$ and $\mu_{\mathcal{E}}(X') \lll\!\!\!\!\!\!\not\;\; \mu_{\mathcal{E}}(Y)$ for all $X' \subsetneq_{\mathcal{E}} X$. Let $\mu_{\mathcal{E}}(Y) = (l_{\mathcal{E}}(Y_1), ..., l_{\mathcal{E}}(Y_m))$ and $\mu_{\mathcal{E}}(X) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_n))$. Assuming that $\mu_{\mathcal{E}}(X) \ggg\!\!\!\!\!\!\not\;\; \mu_{\mathcal{E}}(Y)$, we have one of the following cases:

1. there exists $1 \leq i \leq m$ such that $l_{\mathcal{E}}(Y_k) = l_{\mathcal{E}}(X_k)$ for all $1 \leq k \leq i-1$ and $l_{\mathcal{E}}(Y_i) \gneqq l_{\mathcal{E}}(X_i)$. But we know from Lemma 7.6 that

$$\mu_{\mathcal{E}}(X_i) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_i)),$$

   thus $\mu_{\mathcal{E}}(Y) \lll\!\!\!\!\!\!\not\;\; \mu_{\mathcal{E}}(X_i)$ and we get a contradiction by taking $X' = X_i$.
2. $m \lneqq n$ and $\mu_{\mathcal{E}}(Y)$ is a subword of $\mu_{\mathcal{E}}(X)$. Again by Lemma 7.6 we get

$$\mu_{\mathcal{E}}(Y) = \mu_{\mathcal{E}}(X_m) = (l_{\mathcal{E}}(X_1), ..., l_{\mathcal{E}}(X_m))$$

   which yields a contradiction by choosing $X' = X_m$.

Hence $\mu_{\mathcal{E}}(X) \lll \mu_{\mathcal{E}}(Y)$ and (GR_3) is satisfied. □

7.2. Some basic properties

Krause shows in [16] that the Gabriel-Roiter measure satisfies some properties on abelian length categories, and we are studying here if these properties still hold for finite exact categories. Let $\mu_{\mathcal{E}}$ be the Gabriel-Roiter measure as in Definition 7.2 for the finite exact category $(\mathcal{A}, \mathcal{E})$.

Proposition 7.8. *$\mu_{\mathcal{E}}$ satisfies the following properties:*

(GR_4) *$\mu_{\mathcal{E}}(X) \lll \mu_{\mathcal{E}}(Y)$ or $\mu_{\mathcal{E}}(Y) \lll \mu_{\mathcal{E}}(X)$ for all $X, Y \in ind\mathcal{A}$.*
(GR_5) *$\{\mu_{\mathcal{E}}(X) | X \in ind\,\mathcal{A}, l_{\mathcal{E}}(X) \leq n\}$ is a finite set for all $n \in \mathbb{N}$.*
(GR_6) *$X \in ind\,\mathcal{A}$ is $\mathcal{E}$-simple if and only if $\mu_{\mathcal{E}}(X) \lll \mu_{\mathcal{E}}(Y)$ for all $Y \in ind\,\mathcal{A}$.*

Proof. (GR_4) is clear since $(\mathfrak{S}(\mathbb{N}), \lll)$ is totally ordered. (GR_5) follows from the fact that the set of strictly increasing vectors

$$\{v \in \mathfrak{S}(\mathbb{N}) \,|\, v = \mu_{\mathcal{E}}(X) = (v_1, ..., l_{\mathcal{E}}(X))\}$$

is finite since $l_{\mathcal{E}}(X) \leq n$. To prove (GR_6) we need to remember that $(\mathcal{A}, \mathcal{E})$ is a finite exact category, so all objects are of finite length. Hence each indecomposable object is $\mathcal{E}$-Artinian and thus has an $\mathcal{E}$-simple $\mathcal{E}$-subobject. Let us also note that each indecomposable $\mathcal{E}$-simple object X satisfies $\mu_{\mathcal{E}}(X) = (1)$. □

In the aim to show more properties of Gabriel-Roiter measure on finite exact categories, we extend the following definitions from [16] (3.3) for exact categories:

Definition 7.9. Let $X, Y \in ind\,\mathcal{A}$. We say that X is a *$\mathcal{E}$-Gabriel-Roiter predecessor* of Y if $X \subsetneq_{\mathcal{E}} Y$ and $\mu_{\mathcal{E}}(X) = max_{Y' \subsetneq_{\mathcal{E}} Y} \mu_{\mathcal{E}}(Y')$. An inclusion $X \subsetneq_{\mathcal{E}} Y$ is called *Gabriel-Roiter inclusion* if X is a *Gabriel-Roiter predecessor* of Y. We denote it $X \subset_{\mathcal{E}}^{GR} Y$.

Note that each object $Y \in ind_{\mathcal{A}}$ which is not $\mathcal{E}$-simple admits an $\mathcal{E}$-Gabriel-Roiter predecessor, by (GR_4) and (GR_5). An $\mathcal{E}$-Gabriel-Roiter predecessor X of Y is usually not unique, but the value $\mu_{\mathcal{E}}(X)$ is unique and determined by $\mu_{\mathcal{E}}(Y)$.

Definition 7.10. A chain

$$X_1 \subsetneq_{\mathcal{E}} X_2 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_{n-1} \subsetneq_{\mathcal{E}} X_n = X$$

in $ind\,\mathcal{A}$ is called a *$\mathcal{E}$-Gabriel-Roiter filtration* of X if X_1 is $\mathcal{E}$-simple and X_{i-1} is an *$\mathcal{E}$-Gabriel-Roiter predecessor* of X_i for all $2 \leq i \leq n$.

Proposition 7.11. *A chain*

$$X_1 \subsetneq_{\mathcal{E}} X_2 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_{n-1} \subsetneq_{\mathcal{E}} X_n = X$$

in $ind\,\mathcal{A}$ is an $\mathcal{E}$-Gabriel-Roiter filtration of X if and only if it is a $\mu_{\mathcal{E}}$-filtration of X.

Proof. Let F be a *$\mu_{\mathcal{E}}$-filtration* of X in $ind\mathcal{A}$.

$$F := X_1 \subsetneq_{\mathcal{E}} X_2 \subsetneq_{\mathcal{E}} \cdots \subsetneq_{\mathcal{E}} X_{n-1} \subsetneq_{\mathcal{E}} X_n = X$$

Suppose F is not a Gabriel-Roiter filtration. Then for some $i \in \{1, ..., n-1\}$, X_i is not a $\mathcal{E}$-Gabriel-Roiter predecessor of X_{i+1}, that is, there exists a subobject X' of X_{i+1} such that $\mu_{\mathcal{E}}(X_i) \lll_{r} \mu_{\mathcal{E}}(X')$. Let F' and F_i be filtrations

$$F_i := X_1 \subsetneq_{\mathcal{E}} X_2 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_{i-1} \subsetneq_{\mathcal{E}} X_i$$
$$F' := X'_1 \subsetneq_{\mathcal{E}} X'_2 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X'_{m-1} \subsetneq_{\mathcal{E}} X'_m = X'$$

giving $\mu_{\mathcal{E}}(X_i)$ and $\mu_{\mathcal{E}}(X')$ respectively.

Since both are subobject of X_{i+1}, we can complete both vectors of measure with $l(X_{i+1})$. In this situation, if $\mu_{\mathcal{E}}(X_i)$ is a strict subword of $\mu_{\mathcal{E}}(X')$, then X'_{i+1} being subobject of X_{i+1} gives $(\mu_{\mathcal{E}}(X_i), l(X_{i+1})) \lll\!\!\!\!/ \; (\mu_{\mathcal{E}}(X'), l(X_{i+1}))$.

On the other hand, if $X_j = X'_j$ for all $j \in \{1, ..., l-1\}$ and $l(X'_l) \leq l(X_l)$, then the completion of both vector is trivially order preserving. Both cases lead to a contradiction of F being a $\mu_{\mathcal{E}}$*-filtration*, thus a $\mu_{\mathcal{E}}$*-filtration* is a Gabriel-Roiter filtration.

Conversely, let us show that all Gabriel-Roiter filtrations are $\mu_{\mathcal{E}}$-filtrations. We proceed by induction on m. Of course the Gabriel-Roiter filtrations of length 1 coincide with the $\mu_{\mathcal{E}}$-filtrations of same length. Suppose now that the statement is true for all $l \in \{1, ..., m-1\}$. Let G and F:

$$F := X_1 \subsetneq_{\mathcal{E}} X_2 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} X_{n-1} \subsetneq_{\mathcal{E}} X_n = X$$
$$G := Y_1 \subsetneq_{\mathcal{E}} Y_2 \subsetneq_{\mathcal{E}} ... \subsetneq_{\mathcal{E}} Y_{m-1} \subsetneq_{\mathcal{E}} Y_m = X$$

be two filtrations of X such that F is a $\mu_{\mathcal{E}}$-filtration and G is a Gabriel-Roiter filtration. We know that Y_{m-1} is a Gabriel-Roiter predecessor of X, so $\mu_{\mathcal{E}}(X_{n-1}) \lll \mu_{\mathcal{E}}(Y_{m-1})$. By induction hypothesis, $\mu_{\mathcal{E}}(Y_{m-1}) = (l(Y_1), l(Y_2), ..., l(Y_{m-1}))$ since it is given by a Gabriel-Roiter filtration of Y_{m-1} of length $m-1$. By completing the vector with $l(X)$, using the same reasoning as above, we obtain that

$$(l(X_1), l(X_2), ..., l(X_{n-1}), l(X)) \lll (l(Y_1), l(Y_2), ..., l(Y_{m-1}), l(X)).$$

Since F is a $\mu_{\mathcal{E}}$-filtration, we automatically get

$$(l(X_1), l(X_2), ..., l(X_{n-1}), l(X)) = (l(Y_1), l(Y_2), ..., l(Y_{m-1}), l(X))$$

and thus every Gabriel-Roiter filtration is a $\mu_{\mathcal{E}}$-filtration. $\square$

Proposition 7.12. (GR_7) *Suppose that* $\mu_{\mathcal{E}}(X) \lll\!\!\!\!/ \; \mu_{\mathcal{E}}(Y)$. *Then there are* $Y' \subsetneq_{\mathcal{E}} Y'' \subset_{\mathcal{E}} Y$ *in* ind $\mathcal{A}$ *such that* Y' *is a* $\mathcal{E}$-Gabriel-Roiter predecessor *of* Y'' *with* $\mu_{\mathcal{E}}(Y') \lll \mu_{\mathcal{E}}(X) \lll\!\!\!\!/ \; \mu_{\mathcal{E}}(Y'')$ *and* $l_{\mathcal{E}}(Y') \leq l_{\mathcal{E}}(X)$.

Proof. The proof of (GR_7) in [16] on abelian length categories can be generalized for finite exact categories, we adapt it by replacing each monomorphism by $\subset_{\mathcal{E}}$, and the length function by our length 6.1. $\square$

Now we are studying, always in the more general context of essentially small exact categories, the main property of the Gabriel-Roiter measure due to *Gabriel*, that is shown in [16, 3.4,(GR_8)] for abelian length categories. In fact we will see that it does not always hold for all exact categories.

Definition 7.13. Let $(\mathcal{A}, \mathcal{E})$ be an essentially small exact category. We say that $(\mathcal{A}, \mathcal{E})$ satisfies $(GR8)$ if for each indecomposable object X the following holds: if $X \subset_{\mathcal{E}} Y = \oplus_{i=1}^{r} Y_i$ with indecomposables Y_i, then $\mu_{\mathcal{E}}(X) \lll \max_{1 \leq i \leq r} \mu_{\mathcal{E}}(Y_i)$, and X is a direct summand of Y in case equality holds.

Lemma 7.14. $(GR8)$ *holds for the minimal exact structure* $(\mathcal{A}, \mathcal{E}_{min})$.

Proof. If $X \subset_{\mathcal{E}} Y = \oplus_{i=1}^{r} Y_i$ with respect to the minimal exact structure $\mathcal{E} = \mathcal{E}_{min}$, then X is isomorphic to a direct summand Y_j. Thus $\mu_{\mathcal{E}}(X) \lll \max \mu_{\mathcal{E}}(Y_i)$, and $(GR8)$ holds. $\square$

Remark 7.15. The main property $(GR8)$ holds for the maximal exact structure $\mathcal{E}_{ab}$ when $\mathcal{A}$ is abelian, and for the minimal exact structure $\mathcal{E}_{min}$. However, in general, if we have $X \subset_{\mathcal{E}} Y = \oplus_{i=1}^{r} Y_i$, then $\mu_{\mathcal{E}}(X) = \max \mu_{\mathcal{E}}(Y_i)$ does not always imply that X is a direct summand of Y. We provide an example:

Consider the example discussed in 4.2, and choose the exact structure $\mathcal{E} = \mathcal{E}_3$. If we take $X = S_2$, then X is an $\mathcal{E}$-subobject of $Y = P_1 \oplus P_3$ since we have the Auslander-Reiten sequence $(AR3)$ in $\mathcal{E}$. But all indecomposables are simple in $\mathcal{E}$, so the measure is $\mu_{\mathcal{E}}(X) = \mu_{\mathcal{E}}(P_1) = \mu_{\mathcal{E}}(P_3) = (1)$. That is, even if the condition $\mu_{\mathcal{E}}(X) = \max \mu_{\mathcal{E}}(Y_i)$ is satisfied, X is *not* a direct summand of Y.

This example also illustrates that the property $(GR8)$ is not preserved under reduction: It holds for $(\mathcal{A}, \mathcal{E}_{ab})$ and $(\mathcal{A}, \mathcal{E}_{min})$, but not for the intermediate exact category $(\mathcal{A}, \mathcal{E}_3)$. In general, we do not know which class of exact categories satisfies $(GR8)$.

Let us close this section by the following proposition which modifies the definition of the extension map in [16, 3.4]:

Proposition 7.16. *The Gabriel-Roiter measure 7.2 can be extended to a* measure *defined for all objects in $\mathcal{A}$ (not only the indecomposable ones) as follows:*

$$\mu_{\mathcal{E}} : (Obj\, \mathcal{A}, \subset_{\mathcal{E}}) \to (\mathfrak{S}(\mathbb{N}), \lll)$$
$$X \longmapsto \mu_{\mathcal{E}}(X) = max_{X' \subset_{\mathcal{E}} X}\, \mu_{\mathcal{E}}(X')$$

where $X' \in ind\, \mathcal{A}$ runs through all the indecomposable subobjects of X.

Proof. Clearly, $\mu_{\mathcal{E}}$ is an inclusion-preserving map between the poset $Ojb\mathcal{A}$ with the partial order of 6.11, and $(\mathfrak{S}(\mathbb{N}), \lll)$ which is totally ordered as we have seen above. So $\mu_{\mathcal{E}}$ verifies the condition in 6.13. □

Example 7.17. We revisit Examples 3.5 and 7.3. Consider $X = K^6$, then $\mu_{\mathcal{E}}(K^6) = max\{\mu_{\mathcal{E}_{min}}(K^2), \mu_{\mathcal{E}_{min}}(K^3)\} = (1)$.

8. Basic properties under reduction of exact structures

The aim of this section is to investigate how the basic notions like the $\mathcal{E}$-length and Gabriel-Roiter measure change under reduction of exact structures.

8.1. Reduction of the $\mathcal{E}$-length

Here we prove that the $\mathcal{E}$-length of objects gets reduced when we reduce the corresponding exact structure:

Lemma 8.1. *If $\mathcal{E}$ and $\mathcal{E}'$ are exact structures on $\mathcal{A}$ such that $\mathcal{E}' \subseteq \mathcal{E}$, then $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ for all objects X of $\mathcal{A}$.*

Proof. Let us consider a maximal chain of $\mathcal{E}'$-admissibles monics ending by X

$$0 = X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{n-1}} X_{n-1} \xrightarrow{i_n} X = X_n$$

where $l_{\mathcal{E}'}(X) = n$. Since $\mathcal{E}' \subseteq \mathcal{E}$, all these pairs (i_j, d_j) will also be in $\mathcal{E}$. So the chain above is also a chain of $\mathcal{E}$-admissibles monics and therefore by definition $l_{\mathcal{E}}(X) \geq n$. □

Let us illustrate reduction of length by an example:

Example 8.2. By taking $Ex(\mathcal{A})$ as in 4.2, we re-consider the Example 6.3 and notice that the chain of reductions

$$\mathcal{E}_{min} \subseteq \mathcal{E}_{1,3,5} \subseteq \mathcal{E}_{ab}$$

gives us that

$$l_{\mathcal{E}_{min}}(I_2) = 1 \ < \ l_{\mathcal{E}_{1,3,5}}(I_2) = 2 \ < \ l_{\mathcal{E}_{ab}}(I_2) = 3.$$

8.2. The behavior of $\mathcal{E}$-Gabriel-Roiter measure

We notice that the $\mathcal{E}$-Gabriel-Roiter measure changes in different manners once we reduce the corresponding exact structure. Contrarily to the $\mathcal{E}$-length function, the Gabriel-Roiter measure does not always get reduced; reducing the exact structure could reduce the corresponding Gabriel-Roiter measure and could also enlarge it. We illustrate this situation by some examples:

Example 8.3. We consider $\mathcal{A} = \operatorname{rep} Q$ where

$$Q: \qquad 1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and we consider the following non-split short exact sequences:

$$\begin{array}{rl}
(\mathrm{AR1}) & 0 \longrightarrow 100 \longrightarrow 111 \longrightarrow 011 \longrightarrow 0 \\
(\mathrm{AR2}) & 0 \longrightarrow 001 \longrightarrow 111 \longrightarrow 110 \longrightarrow 0 \\
(\mathrm{AR3}) & 0 \longrightarrow 111 \longrightarrow 011 \oplus 110 \longrightarrow 010 \longrightarrow 0 \\
(4) & 0 \longrightarrow 001 \longrightarrow 011 \longrightarrow 010 \longrightarrow 0 \\
(5) & 0 \longrightarrow 100 \longrightarrow 110 \longrightarrow 010 \longrightarrow 0
\end{array}$$

We construct the exact structures in the same way as in 4.2, and as mentioned in 5.6 the lattice of exact structure $Ex(\mathcal{A})$ is a cube similar to 4.2. The following chains of reduction

$$\mathcal{E}_{min} \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_{1,2} \subseteq \mathcal{E}_{ab}$$
$$\mathcal{E}_{min} \subseteq \mathcal{E}_3 \subseteq \mathcal{E}_{1,3,5} \subseteq \mathcal{E}_{ab}$$

give us for the indecomposable object with dimension vector 111:

$$\mu_{\mathcal{E}_{min}}(111) = (1) \lll \mu_{\mathcal{E}_1}(111) = \mu_{\mathcal{E}_{1,2}}(111) = (1,2) \ggg \mu_{\mathcal{E}_{ab}}(111) = (1,3)$$
$$\mu_{\mathcal{E}_{min}}(111) = \mu_{\mathcal{E}_3}(111) = (1) \lll \mu_{\mathcal{E}_{1,3,5}}(111) = (1,2) \ggg \mu_{\mathcal{E}_{ab}}(111) = (1,3)$$

and for 110

$$\mu_{\mathcal{E}_{min}}(110) = \mu_{\mathcal{E}_3}(110) = (1) \lll \mu_{\mathcal{E}_{1,3,5}}(110) = \mu_{\mathcal{E}_{ab}}(110) = (1,2).$$

So we notice that for this fixed additive category $\mathcal{A} = \operatorname{rep} Q$, by reducing the exact structure $\mathcal{E}$, the corresponding $\mathcal{E}$-Gabriel-Roiter measure gets sometimes reduced, and other times enlarged, even for the same indecomposable object.

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ANNEXE B

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Intersections, sums, and the Jordan-Hölder property for exact categories

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A B S T R A C T

We investigate how the concepts of intersection and sums of subobjects carry to exact categories. We obtain a new characterisation of quasi-abelian categories in terms of admitting admissible intersections in the sense of [23]. There are also many alternative characterisations of abelian categories as those that additionally admit admissible sums and in terms of properties of admissible morphisms. We then define a generalised notion of intersection and sum which every exact category admits. Using these new notions, we define and study classes of exact categories that satisfy the Jordan-Hölder property for exact categories, namely the Diamond exact categories and Artin-Wedderburn exact categories. By explicitly describing all exact structures on $\mathcal{A} = \operatorname{rep}\Lambda$ for a Nakayama algebra Λ we characterise all Artin-Wedderburn exact structures on $\mathcal{A}$ and show that these are precisely the exact structures with the Jordan-Hölder property.

1. Introduction

In a classical theorem in group theory, Camille Jordan stated in 1869 that any two composition series of the same finite group have the same number of quotients. Later, in 1889, Otto Hölder reinforced this result by proving the theorem known as the Jordan-Hölder-Schreier theorem, which states that any two composition series of a given group are equivalent, that is, they have the same length and the same factors, up to permutation and isomorphism. This theorem has been generalised to many other contexts, such as operator groups, modules over rings or general abelian categories. Most proofs use the concept of intersection and sum, which is readily available for groups, modules or objects in an abelian category.

In a general categorical setup, the intersection is defined as pullback of two monomorphisms, if it exists. However, in order to define a sensible cohomology theory, one needs to restrict the notion of subobjects to

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admissible monomorphisms, which allow to form kernel-cokernel pairs. In the context of functional analysis, for instance, this leads to the study of closed subspaces, giving rise to the structure of a quasi-abelian category. More generally, the setup is that of Quillen exact categories [34] which generalises abelian categories. In this generality, one requires not only that the intersection of admissible subobjects exists, but it needs to be an admissible subject itself. Central object of study in this paper is therefore the notion of admissible intersections and sums in an exact category.

The notion of exact categories has been recently the center of many works, see e.g. [25,15–19]. The exact structure for Delta-filtered modules has been studied in [9], and more recently in [27]. They satisfy the Jordan-Hölder property, which is also shown in [37] in the context of stratifying systems in exact categories. Given an exact category, [6] and [20] study its associated Hall algebra, and [41] the graded Lie algebra. Unicity of filtrations for exact categories is also studied in [13] related to the Harder-Narasimhan property for exact categories. And the Jordan-Hölder property in the context of semilattices is studied in [31].

Choosing a Quillen exact structure allows to define various cohomology theories for locally compact abelian groups, Banach spaces, or other categories studied in functional analysis. Other areas where exact structures appear naturally are Happel's construction of triangulated categories from Frobenius categories, or extension-closed subcategories of abelian categories. The set of exact structures on a fixed additive category forms a lattice $(Ex(\mathcal{A}), \subset)$ as shown in [8]. This lattice is studied recently by the first two authors in [3], and also by Fang and Gorsky in [20]. Note also that the exact structures are classified by Enomoto in [15] using Wakamatsu tilting, and in [16] where he gives a classification of all exact structures on a given idempotent complete additive category.

We give now a more detailed description of the main concepts and results in this paper.

Definition 1.1 *(Definition 5.1).* Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A finite $\mathcal{E}$-composition series for an object X of $\mathcal{A}$ is a sequence

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{n-2}}{\rightarrowtail} X_{n-1} \overset{i_{n-1}}{\rightarrowtail} X_n = X$$

where all i_l are *proper admissible monics* with $\mathcal{E}$-simple cokernel. We say an exact category $(\mathcal{A}, \mathcal{E})$ has the *($\mathcal{E}$-)Jordan-Hölder property* or is a *Jordan-Hölder exact category* if any two finite $\mathcal{E}$-composition series of X are equivalent, that is, they have the same length and the same composition factors, up to permutation and isomorphism.

This is an interesting problem since the Jordan-Hölder property does not hold in general for any exact category, see [8, Example 6.9], [18] and Examples 5.3 and 5.12 for counter-examples. This problem is also studied by Enomoto in [18], using the Grothendieck monoid which is a lesser-known invariant of exact categories defined by the same universal property as the Grothendieck group. He shows that the relative Jordan-Hölder property holds if and only if the Grothendieck monoid of the exact category is free. Note that the same author considered the Grothendieck group for exact categories in [17]. In this work we fix an additive category $\mathcal{A}$ and study for which exact structures $\mathcal{E} \in Ex(\mathcal{A})$ the relative $\mathcal{E}$-Jordan-Hölder property holds.

We also establish a generalisation of the Fourth Isomorphism Theorem for modules, which will be a useful tool throughout our work.

Proposition 1.2 *(Proposition 3.8). (**The fourth $\mathcal{E}$-isomorphism theorem**) Let $(\mathcal{A}, \mathcal{E})$ be an exact category and let*

$$X' \rightarrowtail X \twoheadrightarrow X/X'$$

be a short exact sequence in $\mathcal{E}$. Then there is an isomorphism of posets

$$\{M \in \mathcal{A} \mid X' \rightarrowtail M \rightarrowtail X\} \longleftrightarrow \{N \in \mathcal{A} \mid N \rightarrowtail X/X'\}$$
$$M \longmapsto M/X'.$$

In [4], Baumslag gives a short proof of the Jordan-Hölder theorem for groups, by intersecting the terms of one subnormal series with those in another series. Motivated by these ideas, we generalise the abelian notions of intersection and sum to exact categories. We do this in two ways. Firstly, in Section 4, by considering intersections as pullbacks and sums as pushouts of intersections - as is the case in the abelian setting, see [22, Section 5] and [32, Definition 2.6] - we recall *AI-categories* (Admissible Intersection) and *AIS-categories* (Admissible Intersection and Sum) from [23]. The AI-categories are pre-abelian exact categories where admissible monics are stable under pullback along admissible monics and all such pullbacks exist (see Definition 4.1)

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, tail, "g"] \\
C \arrow[r, tail, "j"'] & D.
\end{tikzcd} \tag{1}$$

In a previous version of [23] the term *quasi-nice* exact categories was also used. We prove in Theorem 4.12 that the AI-categories are necessarily quasi-abelian with the maximal exact structure $\mathcal{E}_{max}$ in the lattice $(Ex(\mathcal{A}), \subseteq)$. It has been proved recently by the second author, Shah and Wegner, in [24, Theorem 6.1], that the converse is also true. Hence, we have a new characterisation of quasi-abelian categories:

Theorem 1.3 *(Theorem 4.17).* ***(Brüstle, Hassoun, Shah, Tattar, Wegner)*** *A category $(\mathcal{A}, \mathcal{E}_{max})$ is quasi-abelian if and only if it is an AI-category.*

The AIS-categories are the AI-categories that satisfy the additional property that the unique induced morphism u in the pushout coming from the pullback diagram (1) is an admissible monic (see Definition 4.2):

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, tail, "l"] \arrow[ddr, tail, bend left, "g"] & \\
C \arrow[r, tail, "k"'] \arrow[drr, tail, bend right, "j"'] & E \arrow[dr, tail, "u"] & \\
& & D
\end{tikzcd}$$

It turns out that the AIS-categories are precisely the abelian categories endowed with the maximal exact structure. This, along with our study of the behaviour of admissible morphisms under composition and sum in Section 3.1, allows us to give these following alternative characterisations of abelian categories:

Theorem 1.4 *(Theorems 3.7 and 4.22). Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Then the following are equivalent:*

a) *$\mathcal{A}$ is an abelian category and $\mathcal{E} = \mathcal{E}_{max}$,*
b) *$(\mathcal{A}, \mathcal{E})$ is an AIS-category,*
c) *$Hom(\mathcal{A}) = Hom^{ad}(\mathcal{A})$,*
d) *$Hom^{ad}(\mathcal{A})$ is closed under composition,*
e) *$Hom^{ad}(\mathcal{A})$ is closed under addition,*

where $Hom^{ad}(\mathcal{A})$ denotes the admissible morphisms in $\mathcal{A}$ (see Definition 2.3).

As we observe in Examples 5.3 and 5.4, the pullback and pushout notions of unique intersection and sum do not necessarily apply for general exact categories- even if the exact category has the Jordan Hölder property. This leads us to define, in Definition 5.5, a general notion of admissible intersection and sum that works for all exact categories. For two admissible subobjects (A, f) and (B, g) of X, their intersection, $\mathrm{Int}_X(A, B)$, is the set of all their maximal common proper admissible subobjects. Dually, their sum, $\mathrm{Sum}_X(A, B)$ is the set of all their minimal common proper admissible superobjects that are subobjects of X. Using this, we study exact categories satisfying the Diamond axiom.

Definition 1.5 *(Definition 5.7).* (**Diamond Axiom**) Let $X \in \mathcal{A}$ and let (A, f) and (B, g) be two distinct maximal $\mathcal{E}$-subobjects of X, that is, the cokernels X/A and X/B are $\mathcal{E}$-simple. We say that (A, f) and (B, g) satisfy the *diamond axiom* if for every $Y \in \mathrm{Int}_X(A, B)$ we have that A/Y and B/Y are both $\mathcal{E}$-simple, and are isomorphic to the $\mathcal{E}$-simple cokernels of X and the elements of the sets $\{X/A, A/Y\}\{X/B, B/Y\}$ are equal up to permutation and isomorphism

$$\begin{tikzcd} & A \arrow[dr, tail, "f"] & \\ Y \arrow[ur, tail] \arrow[dr, tail] & & X. \\ & B \arrow[ur, tail, "g"'] & \end{tikzcd}$$

These categories generalise the abelian categories as we note in Remark 5.8, and satisfy the relative Jordan-Hölder property:

Theorem 1.6 *(Theorem 5.11). Every diamond exact category is a Jordan-Hölder exact category.*

Later, in Section 6, we use the new the notion of generalised intersection to define an analog of the Jacobson radical for exact categories, the $\mathcal{E}$*-Jacobson radical*, $\mathrm{rad}_{\mathcal{E}}(X)$, as the generalised intersection of all maximal $\mathcal{E}$-subobjects of X and also introduce the notion of $\mathcal{E}$*-semisimple* objects (see Definitions 6.1 and 6.3). We show some basic properties of the $\mathcal{E}$-Jacobson radical motivated by the properties of the classical Jacobson radical. We then use this to introduce the $\mathcal{E}$*-Artin-Wedderburn categories*, which are exact categories where an analog of the classical Artin-Wedderburn theorem holds:

Definition 1.7 *(Definition 6.4).* An exact category $(\mathcal{A}, \mathcal{E})$ is called *Artin-Wedderburn* if for any object $X \in A$ the following properties are equivalent:

(AW1) Every sequence in $\mathcal{E}$ of the form $A \rightarrowtail X \twoheadrightarrow X/A$ splits,
(AW2) X is $\mathcal{E}$-semisimple,
(AW3) $\mathrm{rad}_{\mathcal{E}}(X) := \mathrm{Int}_X\{(Y, f) \in \mathcal{S}_X \mid \mathrm{Coker}\, f \text{ is } \mathcal{E} - \text{simple}\} = \{0\}$.

Here $\mathcal{S}_X$ is the poset of all *proper* $\mathcal{E}$-subobjects of X (see Definition 2.22). We call in this case $\mathcal{E}$ an *Artin-Wedderburn* exact structure on $\mathcal{A}$.

We give examples of such categories and prove in Lemma 6.7, that every additive category with the minimal exact structure $\mathcal{E}_{min}$ in the lattice $(Ex(\mathcal{A}), \subseteq)$; the split exact structure, is an $\mathcal{E}$-Artin-Wedderburn category. Then, by showing that certain $\mathcal{E}$-Artin-Wedderburn categories satisfy the Diamond axiom, we obtain the following result:

Theorem 1.8 *(Theorem 6.8). Let $(\mathcal{A}, \mathcal{E})$ be a Krull-Schmidt $\mathcal{E}$-Artin-Wedderburn category. Then $(\mathcal{A}, \mathcal{E})$ is a Jordan-Hölder exact category.*

We then give for any Nakayama algebra, Λ, an explicit description of all exact structures on $\operatorname{rep}\Lambda$ in Theorem 6.9 and use this to characterise all Artin-Wedderburn exact structures on $\operatorname{rep}\Lambda$ in Theorem 6.11. It turns out these they are exactly the Jordan-Hölder exact structures on $\operatorname{rep}\Lambda$:

Theorem 1.9 *(Theorem 6.12). Let Λ be a Nakayama algebra, and denote $\mathcal{A} = mod\,\Lambda$, the category of finitely generated left Λ-modules. Then an exact category $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artin-Wedderburn precisely when it is Jordan-Hölder.*

Once satisfied, the $\mathcal{E}$-Jordan-Hölder property allows to define *the $\mathcal{E}$-Jordan-Hölder length function* (compare also [18, 4.1]):

Definition 1.10 *(Definition 7.1).* The $\mathcal{E}$-Jordan-Hölder length $l_{\mathcal{E}}(X)$ of an object X in $\mathcal{A}$ is the length of an $\mathcal{E}$-composition series of X. That is $l_{\mathcal{E}}(X) = n$ if and only if there exists an $\mathcal{E}$-composition series

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X.$$

We say in this case that X is $\mathcal{E}$-finite.

This $\mathcal{E}$-Jordan-Hölder length function has good properties that improves the general length defined and studied on any exact category in [8, Definition 6.1, Theorem 6.6] in which there is only an inequality.

Proposition 1.11 *(Corollary 7.2). Let $X \rightarrowtail Z \twoheadrightarrow Y$ be an admissible short exact sequence of finite length objects. Then*

$$l_{\mathcal{E}}(Z) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y).$$

Moreover this length function satisfies also important properties as:

Proposition 1.12 *(Proposition 7.5). (**$\mathcal{E}$-Hopkins-Levitzki theorem**) An object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian if and only if it has an $\mathcal{E}$-finite length.*

Finally, the $\mathcal{E}$-Jordan Hölder length function can only decrease under reduction of exact structures:

Proposition 1.13 *(Proposition 7.8). If $\mathcal{E}$ and $\mathcal{E}'$ are exact structures on $\mathcal{A}$ such that $\mathcal{E}' \subseteq \mathcal{E}$, then $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ for all objects X in $\mathcal{A}$.*

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2. Background

In this section we recall from [22,10] the definition of a Quillen exact structure along with the definitions of various types of additive categories and other important concepts that form the backdrop to our work.

2.1. Exact structures on additive categories

Definition 2.1. An *additive category* $\mathcal{A}$ is a preadditive category (all its hom-sets are abelian groups and composition of morphisms is bilinear) admitting all finite biproducts.

Definition 2.2. Let $\mathcal{A}$ be an additive category. A kernel-cokernel pair (i, d) in $\mathcal{A}$ is a pair of composable morphisms such that i is kernel of d and d is cokernel of i. If a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathcal{A}$ is fixed, an *admissible monic* is a morphism i for which there exist a morphism d such that $(i, d) \in \mathcal{E}$. An *admissible epic* is defined dually. Note that admissible monics and admissible epics are referred to as inflation and deflation in [22], respectively. We depict an admissible monic by $\rightarrowtail$ and an admissible epic by $\twoheadrightarrow$. An *exact structure* $\mathcal{E}$ on $\mathcal{A}$ is a class of kernel-cokernel pairs (i, d) in $\mathcal{A}$ which is closed under isomorphisms and satisfies the following axioms:

- (A0) For all objects $A \in Obj\mathcal{A}$ the identity 1_A is an admissible monic,
- $(\mathrm{A0})^{op}$ For all objects $A \in Obj\mathcal{A}$ the identity 1_A is an admissible epic,
- (A1) The class of admissible monics is closed under composition,
- $(\mathrm{A1})^{op}$ The class of admissible epics is closed under composition,
- (A2) The pushout of an admissible monic $i : A \rightarrowtail B$ along an arbitrary morphism $f : A \to C$ exists and yields an admissible monic j:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "g"] \\ C \arrow[r, tail, "j"'] & D, \end{tikzcd}$$

- $(\mathrm{A2})^{op}$ The pullback of an admissible epic h along an arbitrary morphism g exists and yields an admissible epic k

$$\begin{tikzcd} A \arrow[r, two heads, "k"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "g"] \\ C \arrow[r, two heads, "h"'] & D. \end{tikzcd}$$

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ consisting of an additive category $\mathcal{A}$ and an exact structure $\mathcal{E}$ on $\mathcal{A}$. The pairs (i, d) forming the class $\mathcal{E}$ are called *admissible short exact sequences*, *$\mathcal{E}$-sequences*, or just *admissible sequences*.

Definition 2.3. [10, Definition 8.1] A morphism $f : A \to B$ in an exact category is called admissible if it factors as $f = me$ where m is an admissible monic and e is an admissible epic. Admissible morphisms will sometimes be displayed as

$$A \overset{f}{-\!\!\circ\!\!\to} B$$

in diagrams, and the classes of admissible arrows of $\mathcal{A}$ will be denoted as $\mathrm{Hom}^{ad}_{\mathcal{A}}(-,-)$.

Proposition 2.4. *[10, Proposition 2.16] Suppose that $i : A \to B$ is a morphism in $\mathcal{A}$ admitting a cokernel. If there exists a morphism $j : B \to C$ such that the composition $j \circ i : A \rightarrowtail C$ is an admissible monic, then i is an admissible monic.*

Definition 2.5. An additive category $\mathcal{A}$ is *pre-abelian* if it has kernels and cokernels.

Remark 2.6. Let $\mathcal{A}$ be a *pre-abelian* category, then it admits pullbacks and pushouts.

Definition 2.7. [35, page 524] A kernel (A, f) is called *semi-stable* if for every existing pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "s_B"] \\ C \arrow[r, "s_C"'] & S \end{tikzcd}$$

the morphism s_C is also a kernel. We define dually a *semi-stable* cokernel. A short exact sequence $A \overset{i}{\rightarrowtail} B \overset{d}{\twoheadrightarrow} C$ is said to be *stable* if i is a semi-stable kernel and d is a semi-stable cokernel. We denote by $\mathcal{E}_{sta}$ the class of all *stable* short exact sequences.

Remark 2.8. [38, Theorem 3.3] In a pre-abelian category $\mathcal{A}$, the class $\mathcal{E}_{sta}$ forms an exact structure.

Definition 2.9. Let $\mathcal{A}$ be a pre-abelian category. A morphism f is called *strict* if the canonical map $\bar{f}$ (which is the unique morphism that exists between the image and the coimage of f) is an isomorphism: $Coim(f) \cong Im(f)$.
A short exact sequence $A \overset{i}{\rightarrowtail} B \overset{d}{\twoheadrightarrow} C$ is said *strict* if i is strict or d is strict. We denote by $\mathcal{E}_{str}$ the class of all *strict* short exact sequences.
The class $\mathcal{E}_{str}$ defines the maximal exact structure $\mathcal{E}_{max} = \mathcal{E}_{str}$ in any pre-abelian category, as shown in [38].

Remark 2.10. We denote by $\mathcal{E}_{all}$ the *class* of all short exact sequences in an additive category $\mathcal{A}$. We use the notation $\mathcal{E}_{all}$ despite the fact that the class $\mathcal{E}_{all}$ does not necessarily form an exact structure in an additive category $\mathcal{A}$.

Definition 2.11. An additive category $\mathcal{A}$ is *quasi-abelian* if it is *pre-abelian* and $\mathcal{E}_{all} = \mathcal{E}_{sta}$.

It is clear that an additive category $\mathcal{A}$ is quasi-abelian if it is pre-abelian and every pullback of a strict epimorphism is a strict epimorphism, and every pushout of a strict monomorphism is a strict monomorphism.

Definition 2.12. An additive category $\mathcal{A}$ is *abelian* if and only if it is *pre-abelian* and $\mathcal{E}_{all} = \mathcal{E}_{str}$.

Hence abelian categories with their maximal exact structure $(\mathcal{A}, \mathcal{E}_{all})$ are the pre-abelian additive categories where every morphism is admissible.

It is well known that the class of all split short exact sequences forms an exact structure on every additive category, called the minimal exact structure $\mathcal{E}_{min}$. Note that certain properties of the underlying additive category $\mathcal{A}$ determine which exact structures can exist on $\mathcal{A}$. See [8, Section 2] for a summary on the minimal and maximal exact structures on any additive category. Moreover, under some finiteness conditions, the exact structures on $\mathcal{A}$ are parametrized by subsets of Auslander-Reiten sequences. This phenomenon was observed in [8, Theorem 5.7], and is based on [17]:

Theorem 2.13. *Let $\mathcal{A}$ be a skeletally small, Hom-finite, idempotent complete additive category which has finitely many indecomposable objects up to isomorphism. Then every exact structure $\mathcal{E}$ on $\mathcal{A}$ is uniquely determined by the set $\mathcal{B}$ of Auslander-Reiten sequences that are contained in $\mathcal{E}$. We write in this case $\mathcal{E} = \mathcal{E}(\mathcal{B})$.*

2.2. The poset of $\mathcal{E}$-subobjects

Now let us also recall the following useful and well known notions:

Definition 2.14. A poset P is called a *lattice* if for every pair of elements of P there exists a supremum and an infimum. In other words, there exist two binary operations $\vee$ and $\wedge$: $P \times P \to P$ satisfying the following axioms:

1. $\vee$ is associative and commutative,
2. $\wedge$ is associative and commutative,
3. $\wedge$ and $\vee$ satisfy the following property:

$$m \vee (m \wedge n) = m = m \wedge (m \vee n) \text{ for all } m, n \in P.$$

Remark 2.15. As a consequence of the axioms above we have the following property for lattices:

$$m \vee m = m \text{ and } m \wedge m = m \text{ for all } m \in P.$$

Definition 2.16. A lattice $(P, \leq, \wedge, \vee)$ is *modular* if the following property is satisfied for all $x_1, x_2, x_3 \in P$ with $x_1 \leq x_2$:

$$x_2 \wedge (x_1 \vee x_3) = x_1 \vee (x_2 \wedge x_3).$$

Definition 2.17. [8, Definition 3.1] Let A and B be objects of an exact category $(\mathcal{A}, \mathcal{E})$. If there is an admissible monic $i : A \rightarrowtail B$ we say the pair (A, i) is an *admissible subobject* or *$\mathcal{E}$-subobject of* B. Often we will refer to the pair (A, i) by the object A and write $A \subset_{\mathcal{E}} B$. If i is not an isomorphism, we use the notation $A \subsetneq_{\mathcal{E}} B$ and if, in addition, $A \ncong 0$ we say that (A, i) is a *proper* admissible subobject of B.

Definition 2.18. [8, Definition 3.3] A non-zero object S in $(\mathcal{A}, \mathcal{E})$ is *$\mathcal{E}$-simple* if S admits no $\mathcal{E}$-subobjects except 0 and S, that is, whenever $A \subset_{\mathcal{E}} S$, then A is the zero object or isomorphic to S.

Remark 2.19. Let A be an $\mathcal{E}$-subobject of B given by the monic $i : A \rightarrowtail B$. We denote by $B/^i A$ (or simply B/A when i is clear from the context) the cokernel of i, thus we denote the corresponding admissible sequence as

$$A \overset{i}{\rightarrowtail} B \twoheadrightarrow B/A$$

Remark 2.20. An admissible monic $i : A \rightarrowtail B$ is proper precisely when its cokernel is non-zero. In fact, by uniqueness of kernels and cokernels, the exact sequence

$$B \overset{1_B}{\rightarrowtail} B \twoheadrightarrow 0$$

is, up to isomorphism, the only one with zero cokernel. Thus an admissible monic i has $\mathrm{Coker}\,(i) = 0$ precisely when i is an isomorphism. Dually, an admissible epic $d : B \twoheadrightarrow C$ is an isomorphism precisely when $\mathrm{Ker}\,(d) = 0$. In particular a morphism which is at the same time an admissible monic and epic is an isomorphism.

Definition 2.21. [8, Section 6.1] An object X of $(\mathcal{A}, \mathcal{E})$ is *$\mathcal{E}$-Noetherian* if any increasing sequence of $\mathcal{E}$-subobjects of X

$$X_1 \rightarrowtail X_2 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n \rightarrowtail X_{n+1} \dots$$

becomes stationary. Dually, an object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$*-Artinian* if any descending sequence of $\mathcal{E}$-subobjects of X

$$\dots X_{n+1} \rightarrowtail X_n \rightarrowtail X_{n-1} \rightarrowtail \dots \rightarrowtail X_2 \rightarrowtail X_1$$

becomes stationary. An object X which is both $\mathcal{E}$-Noetherian and $\mathcal{E}$-Artinian is called $\mathcal{E}$*-finite*. The exact category $(\mathcal{A}, \mathcal{E})$ is called $\mathcal{E}$*-Artinian* (respectively $\mathcal{E}$*-Noetherian*, $\mathcal{E}$*-finite*) if every object is $\mathcal{E}$-Artinian (respectively $\mathcal{E}$-Noetherian, $\mathcal{E}$-finite).

Now let us recall a definition similar to [18, Definition 2.1]:

Definition 2.22. Two $\mathcal{E}$-subobjects ($Y_i \overset{f_i}{\rightarrowtail} X$) for $i = 0, 1$ are *isomorphic* $\mathcal{E}$*-subobjects of* X if there exists an isomorphism $\phi \in \mathcal{A}(Y_0, Y_1)$ such that $f_0 = f_1 \circ \phi$. We denote by $\mathcal{P}_X^{\mathcal{E}}$ the set of isomorphism classes of $\mathcal{E}$-subobjects of X. The relation

$$(Y, f) \leq (Z, g) \Longleftrightarrow \exists \begin{tikzcd} Y \arrow[r, tail, "\exists h"] \arrow[dr, tail, "f"'] & Z \arrow[d, tail, "g"] \\ & X \end{tikzcd}$$

turns $(\mathcal{P}_X^{\mathcal{E}}, \leq)$ into a poset. Sometimes, to avoid clutter, we will drop the superscript $\mathcal{E}$ and write $\mathcal{P}_X$. By $\mathcal{S}_X$ we denote the set of isomorphism classes of proper $\mathcal{E}$-subobjects of X, thus $\mathcal{P}_X^{\mathcal{E}} = \mathcal{S}_X \cup \{0\} \cup \{X\}$ and $\mathcal{S}_X$ inherits a poset structure from $\mathcal{P}_X$.

Remark 2.23. An $\mathcal{E}$-subobject (Y, f) of X is a maximal element of $\mathcal{P}_X^{\mathcal{E}}$ if and only if Coker f is $\mathcal{E}$-simple. For a poset $(P, \leq)$, by $\mathrm{Max}(P)$ we denote the maximal elements of the poset. Thus $\mathrm{Max}(\mathcal{S}_X)$ is the class of *maximal* $\mathcal{E}$*-subobjects* of X. Note also that an object X is $\mathcal{E}$-finite precisely when the length of all chains in the poset $\mathcal{P}_X^{\mathcal{E}}$ is bounded.

3. General results

We show an $\mathcal{E}$-version of the fourth isomorphism theorem. We also give some results describing the behaviour of admissible morphisms, which yields a new characterisation of abelian categories in Theorem 3.7.

3.1. Admissible morphisms and abelian categories

In this subsection we show that the admissible morphisms in an exact category behave poorly, unless we work in an abelian category with the maximal exact structure. Let us first recall the following related results:

Proposition 3.1. *[23, Lemma 3.5] (***The** $\mathcal{E}$**-Schur lemma***) Let* $X \overset{f}{\multimap} Y$ *be an admissible non-zero morphism. Then the following hold:*

a) *if* X *is* $\mathcal{E}$*-simple, then* f *is an admissible monic,*
b) *if* Y *is* $\mathcal{E}$*-simple, then* f *is an admissible epic.*

Corollary 3.2. *[23, Corollary 3.6] Let* S *be an* $\mathcal{E}$*-simple object, then the non-zero admissible endomorphisms* $S \overset{f}{\multimap} S$ *form the group* $Aut(S)$ *of automorphisms of* S*.*

Remark 3.3. The classical Schur lemma on abelian categories states that the endomorphism ring of a simple object is a division ring. We show in Corollary 3.2 that any non-zero admissible endomorphism of an $\mathcal{E}$-simple object is invertible, but it is not true in general that the set of admissible endomorphisms forms a ring. In fact, the composition of admissible morphisms needs not be admissible, (see [10, Remark 8.3]), nor is it true for sums of admissible morphisms, as we discuss in this section.

The following fact will be our main tool:

Lemma 3.4. *[21, Proposition 3.1] Suppose that every morphism in $\mathcal{A}$ is admissible, then $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{max} = \mathcal{E}_{all}$.*

Lemma 3.5. *Suppose that the class of admissible morphisms in $\mathcal{A}$ is closed under composition. Then $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{max} = \mathcal{E}_{all}$.*

Proof. We show that every morphism can be written as the composition of a section followed by a retraction. Whence the claim will follow from Lemma 3.4 since sections and retractions are always admissible morphisms, since the split exact structure is the minimal exact structure on any additive category. To this end, let $f : X \to Y$ be an arbitrary morphism in $\mathcal{A}$ and consider the two split short exact sequences

$$X \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} X \oplus Y \xrightarrow{[0\ 1]} Y$$

$$X \xrightarrow{\left[\begin{smallmatrix}1\\f\end{smallmatrix}\right]} X \oplus Y \xrightarrow{[-f\ 1]} Y.$$

Then there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "\left[\begin{smallmatrix}1\\f\end{smallmatrix}\right]"'] & & Y \\ & X \oplus Y \arrow[ur, two heads, "{[0\ 1]}"'] & \end{tikzcd}$$

which proves the claim. □

Lemma 3.6. *Suppose that the class of admissible morphisms in $\mathcal{A}$ is closed under addition. Then $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{max} = \mathcal{E}_{all}$.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Then

$$\begin{bmatrix}0\\f\end{bmatrix} = \begin{bmatrix}1\\f\end{bmatrix} + \begin{bmatrix}-1\\0\end{bmatrix} : X \to X \oplus Y$$

is the sum of two sections and is hence admissible by assumption. Let

$$\begin{tikzcd} X \arrow[rr, "\left[\begin{smallmatrix}0\\f\end{smallmatrix}\right]"] \arrow[dr, two heads, "g"'] & & X \oplus Y \\ & Z \arrow[ur, tail, "\left[\begin{smallmatrix}h'\\h\end{smallmatrix}\right]"'] & \end{tikzcd}$$

be a factorisation of $\left[\begin{smallmatrix}0\\f\end{smallmatrix}\right]$ into an admissible epic followed by an admissible monic. Observe that, as g is epic, $h' = 0$. We claim that if $\left[\begin{smallmatrix}0\\h\end{smallmatrix}\right]$ is an admissible monic then so is h, whence $f = hg$ and is therefore admissible from which the statement will follow from Lemma 3.4. Observe that Coker $\left[\begin{smallmatrix} 0 \\ h \end{smallmatrix}\right] \cong X \oplus \operatorname{Coker} h$ and consider the pullback of short exact sequences

$$\begin{tikzcd} Z \arrow[r, tail, "\alpha"] \arrow[d, equal] & P \arrow[r, two heads, "\beta"] \arrow[d, "\left[\begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix}\right]"] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Coker} h \arrow[d, "\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]"] \\ Z \arrow[r, tail, "\left[\begin{smallmatrix} 0 \\ h \end{smallmatrix}\right]"'] & X \oplus Y \arrow[r, two heads, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & \operatorname{coker} h \end{smallmatrix}\right]"'] & X \oplus \operatorname{Coker} h. \end{tikzcd}$$

It is straightforward to verify that $P \cong Y$, $\beta = \operatorname{coker} h$, $\left[\begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $\alpha = h$. Thus we are done. $\square$

This shows that, in general, the set of admissible endomorphisms $\operatorname{End}^{ad}_{\mathcal{A}}(X)$ is not a subring of $\operatorname{End}_{\mathcal{A}}(X)$ under the usual addition and composition, also that $\operatorname{Hom}^{ad}(X, Y)$ is not a group under the usual addition. To finish, we summarise the results of this subsection.

Theorem 3.7. *The following conditions are equivalent:*

a) *$\mathcal{A}$ is an abelian category,*
b) *$Hom(\mathcal{A}) = Hom^{ad}(\mathcal{A})$,*
c) *$Hom^{ad}(\mathcal{A})$ is closed under composition,*
d) *$Hom^{ad}(\mathcal{A})$ is closed under addition.*

Proof. We know that in an abelian category $\mathcal{A}$ every morphism is admissible so $\operatorname{Hom}(\mathcal{A}) = \operatorname{Hom}^{ad}(\mathcal{A})$ and it is closed under the composition and the addition of the category $\mathcal{A}$.
The converse is follows from Lemmas 3.4, 3.5 and 3.6. $\square$

3.2. Isomorphism theorem

We give a generalisation of the fourth isomorphism theorem for modules to exact categories:

Proposition 3.8. ***(The fourth $\mathcal{E}$-isomorphism theorem)*** *Let $(\mathcal{A}, \mathcal{E})$ be an exact category and let*

$$X' \rightarrowtail X \twoheadrightarrow X/X'$$

be a short exact sequence in $\mathcal{E}$. Then there is an isomorphism of posets

$$\{M \in \mathcal{A} \mid X' \rightarrowtail M \rightarrowtail X\} \longleftrightarrow \{N \in \mathcal{A} \mid N \rightarrowtail X/X'\} = \mathcal{P}^{\mathcal{E}}_{X/X'}$$
$$M \longmapsto M/X'.$$

Proof. Let us begin by showing that the correspondence is bijective. First we note that the map $M \mapsto M/X'$ is well-defined by [10, Lemma 3.5]. Next, we define an inverse map ϕ. For $N \rightarrowtail X/X'$ define $\phi(N)$ to be the pullback

$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d, equal] & \phi(N) \arrow[r, two heads] \arrow[d, "\alpha"] \arrow[dr, phantom, "\lrcorner"] & N \arrow[d, tail] \\ X' \arrow[r, tail] & X \arrow[r, two heads] & X/X'. \end{tikzcd}$$

We observe that by [10, Proposition 2.15], α is an admissible monic and thus ϕ is a well-defined map. We now show that the maps are mutually inverse. For $X' \rightarrowtail M \rightarrowtail X$ the fact that $\phi(M/X') \cong M$ follows from applying the Five Lemma for exact categories [10, Corollary 3.2] to the diagram

$$\begin{tikzcd}
X' \arrow[r, tail] \arrow[d, equal] & \phi(M/X') \arrow[r, two heads] \arrow[d] & M/X' \arrow[d, equal] \\
X' \arrow[r, tail] & M \arrow[r, two heads] & M/X'.
\end{tikzcd}$$

For $N \rightarrowtail X/X'$, there is a short exact sequence

$$X' \rightarrowtail \phi(N) \twoheadrightarrow N.$$

Thus, $\phi(N)/X' \cong N$ and we are done.

Now we show that this is an isomorphism of posets. First we show that if $X' \rightarrowtail M' \rightarrowtail M \rightarrowtail X$ then $M'/X' \rightarrowtail M/X'$. This follows from applying [10, Lemma 3.5] to the diagram

$$\begin{tikzcd}
X' \arrow[r, tail] \arrow[d, equal] & M' \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & M'/X' \arrow[d] \\
X' \arrow[r, tail] & M \arrow[r, two heads] & M/X'.
\end{tikzcd}$$

Finally, we show the converse, that is if $M'/X' \rightarrowtail M/X' \rightarrowtail X/X'$ then $M' \rightarrowtail M$. From earlier in the proof, there is a commutative diagram

$$\begin{tikzcd}
M' \arrow[r, tail] \arrow[d, "\alpha"'] & M'/X' \arrow[d, tail] \\
M \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & M/X' \arrow[d, tail] \\
X \arrow[r, two heads] & X/X',
\end{tikzcd}$$

with the outer rectangle being a pullback. Thus, by the Pullback Lemma and [10, Proposition 2.15], α is an admissible monic. $\square$

Remark 3.9. By the Fourth $\mathcal{E}$-isomorphism theorem (Proposition 3.8), an $\mathcal{E}$-subobject (Y, f) of an object X is $\mathcal{E}$-maximal if and only if for all commutative diagrams

$$\begin{tikzcd}
Y \arrow[d, tail, "g"'] \arrow[dr, tail, "f"] & \\
Z \arrow[r, tail, "h"'] & X
\end{tikzcd}$$

either g or h is an isomorphism.

4. The AI and AIS exact categories

In abelian categories, the notions of intersection and sum of subobjects are given by pullbacks and pushouts respectively, see [22, Section 5] and [32, Definition 2.6]. In this paragraph, we investigate these concepts carry to exact categories. We recall the definitions of admissible intersection and sum that first defined by the second author in [23], then show that these lead to characterisations of quasi-abelian and abelian categories respectively.

4.1. Definitions and properties

The intersection, which exists and is well defined in a pre-abelian exact category, is not necessarily an *admissible* subobject. We recall the definition of exact categories satisfying the admissible intersection property and the admissible sum property from [23]. Note that, in a previous version of [23], the name quasi-n.i.c.e. was used in the sense that they are **n**ecessarily **i**ntersection **c**losed **e**xact categories, and which we will call **A.I** since they admit **A**dmissible **I**ntersections:

Definition 4.1. [23, Definition 4.3] An exact category $(\mathcal{A}, \mathcal{E})$ is called an *AI-category* if $\mathcal{A}$ is pre-abelian additive category satisfying the following additional axiom:

(AI) The pullback A of two admissible monics $j : C \rightarrowtail D$ and $g : B \rightarrowtail D$ exists and yields two admissible monics i and f.

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, tail, "f"'] & B \arrow[d, tail, "g"] \\ C \arrow[r, tail, "j"'] & D \end{tikzcd}$$

Let us now introduce a special sub-class of the AI exact categories, that we call **A.I.S** exact categories, since they admit **A**dmissible **I**ntersections and **S**ums:

Definition 4.2. [23, Definition 4.5] An exact category $(\mathcal{A}, \mathcal{E})$ is called an *AIS-category* if it is an AI-category and moreover it satisfies the following additional axiom:

(AS) The morphism u in the diagram below, given by the universal property of the pushout E of i and f coming from the pullback diagram of the axiom (AIS) above, is an admissible monic.

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, tail, "f"'] & B \arrow[d, tail, "l"] \arrow[ddr, tail, bend left, "g"] & \\ C \arrow[r, tail, "k"'] \arrow[drr, tail, bend right, "j"'] & E \arrow[dr, tail, "u"] & \\ & & D \end{tikzcd}$$

Remark 4.3. One may consider the duals of the above definitions by taking admissible epics instead of monics. Since our focus is on $\mathcal{E}$-subobjects we only study the above and simply remark that the dual definitions lead to the duals of the results of the rest of Section 4, which hold without statement.

Assume now that $(\mathcal{A}, \mathcal{E})$ is an AIS-category and let us define relative notions of intersection and sum:

Definition 4.4. [23, Definition 4.6] Let (X_1, i_1), (X_2, i_2) be two $\mathcal{E}$-subobjects of an object X. We define their *intersection* $X_1 \cap_X X_2$, to be the pullback

$$\begin{tikzcd} X_1 \cap_X X_2 \arrow[r, tail, "s_1"] \arrow[d, tail, "s_2"'] & X_1 \arrow[d, tail, "i_1"] \arrow[dl, phantom, "\lrcorner" very near end] \\ X_2 \arrow[r, tail, "i_2"'] & X. \end{tikzcd}$$

We then define their *sum*, $X_1 +_X X_2$, to be the pushout

$$\begin{tikzcd} X_1 \cap_X X_2 \arrow[r, tail, "s_1"] \arrow[d, tail, "s_2"'] \arrow[dr, phantom, "\ulcorner" very near start] & X_1 \arrow[d, tail, "j_1"] \\ X_2 \arrow[r, tail, "j_2"'] & X_1 +_X X_2. \end{tikzcd}$$

Remark 4.5. Equivalently, for two $\mathcal{E}$-subobjects (X_1, i_1), (X_2, i_2) of an object X we have

$$X_1 \cap_X X_2 = \operatorname{Ker}\left(X_1 \oplus X_2 \xrightarrow{[i_1 - i_2]} X \right)$$

and

$$X_1 +_X X_2 = \operatorname{Coker}\left(X_1 \cap_X X_2 \xrightarrow{[s_1 - s_2]^t} X_1 \oplus X_2 \right).$$

Thus, as the direct sum is an associative operation, so are the sum and intersection operations. Moreover, the direct sum is commutative up to isomorphism, and so are the sum and intersection.

Let us note that these definitions generalise the abelian versions as shown in [23].

Remark 4.6. [23, 4.8, 4.12, 4.13] Let (X_1, i_1), (X_2, i_2) and (Y, j) be $\mathcal{E}$-subobjects of an object X. Then

a) $X_1 \cap_X X_1 = X_1 = X_1 +_X X_1$.
b) If $X_1 +_X X_2 = 0_{\mathcal{A}}$ then $X_1 = X_2 = 0_{\mathcal{A}}$.
c) If $(\mathcal{A}, \mathcal{E})$ is an AI-category and there exists an admissible monic

$$i : X_1 \rightarrowtail X_2$$

then there exists an admissible monic

$$X_1 \cap Y \rightarrowtail X_2 \cap Y.$$

d) If $(\mathcal{A}, \mathcal{E})$ is an AIS-category and there exists an admissible monic

$$i : X_1 \rightarrowtail X_2$$

then there exists an admissible monic

$$X_1 + Y \rightarrowtail X_2 + Y.$$

Lemma 4.7. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category and let $f : X \rightarrowtail Z$ and $g : Y \rightarrowtail Z$ be admissible monics. Suppose that $X \cap_Z Y$ exists and is the zero object, then $X +_Z Y \cong X \oplus Y$.*

Proof. By assumption, there is a pullback diagram in $\mathcal{A}$:

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & X \arrow[d, tail, "f"] \\ Y \arrow[r, tail, "g"'] \arrow[ur, phantom, "\lrcorner"] & Z. \end{tikzcd}$$

By direct computation we have that

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, tail, "s"] \\ Y \arrow[r, tail, "t"'] & X \oplus Y \end{tikzcd}$$

is a pushout diagram for any pair of morphisms s and t satisfying the universal property of the coproduct. Thus, by definition, $X +_Z Y \cong X \oplus Y$. □

4.2. AI-categories and quasi-abelian categories

It is not difficult to see that the split exact structure $\mathcal{E}_{min}$ does *not* satisfy axiom (AI) unless every sequence splits in $\mathcal{A}$. Compare also [26, Remark 2.4 and 5.3] which helps to show that the category of abelian groups equipped with $\mathcal{E}_{min}$ does *not* satisfy axiom (AI). In fact, an exact structure needs to contain *all* short exact sequences in order to satisfy the (AI) axiom:

Proposition 4.8. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. If $(A, \mathcal{E})$ is an AI-category, then $\mathcal{E} = \mathcal{E}_{all}$.*

Proof. Let us suppose that the exact structure $\mathcal{E}$ is strictly included in $\mathcal{E}_{all}$, thus there exists a short exact sequence

$$S: \qquad 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

such that $S \notin \mathcal{E}$.

Consider the two sections

$$\begin{bmatrix} 1 \\ g \end{bmatrix} : M \to M \oplus N$$

and

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} : M \to M \oplus N.$$

It is easy to verify that the pull-back of these two morphisms is:

$$\begin{tikzcd} L \arrow[r, tail, "f"] \arrow[d, tail, "f"'] & M \arrow[d, tail, "{\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right]}"] \\ M \arrow[r, tail, "{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}"'] \arrow[ur, phantom, "\lrcorner"] & M \oplus N. \end{tikzcd}$$

Since f is not admissible in $\mathcal{E}$, the (AI) axiom is not satisfied and $(\mathcal{A}, \mathcal{E})$ is therefore not an AI-category. □

Remark 4.9. The previous proposition shows that an exact structure satisfying the (AI) axiom is unique, when it exists on an additive category.

Before showing that the AI categories form a sub-class of quasi-abelian additive categories, let us recall the following:

Lemma 4.10. *[39, 1.1.7][10, 4.4] In any quasi-abelian category, the class of all short exact sequences defines an exact structure $\mathcal{E}_{all}$ and this is the maximal one $\mathcal{E}_{max} = \mathcal{E}_{all}$. In particular this is the case for abelian categories (see also [36]).*

Lemma 4.11. *Every additive category $\mathcal{A}$ admitting $\mathcal{E}_{all}$ as an exact structure is quasi-abelian.*

Proof. By the axioms (A2) and (A2)^{op} of an exact structure, every short exact sequence is stable. So it follows from Definition 2.11 that $\mathcal{A}$ is quasi-abelian. □

Theorem 4.12. *Every AI-category $\mathcal{A}$ is quasi-abelian.*

Proof. By Proposition 4.8, every exact structure satisfying the (AI) axiom is $\mathcal{E} = \mathcal{E}_{all}$ and then, by Lemma 4.11 $\mathcal{A}$, is quasi-abelian. □

Example 4.13. We provide an example which demonstrates that not every quasi-abelian additive category with its maximal exact structure admits admissible sums: Consider the quiver

$$Q: \qquad 1 \longrightarrow 2 \longrightarrow 3$$

The Auslander-Reiten quiver of $\operatorname{rep} Q$ is as follows:

$$\begin{tikzcd}
 & & P_1 \arrow[dr] & & \\
 & P_2 \arrow[ur, "a"] \arrow[dr, "b"'] & & I_2 \arrow[ll, dotted] \arrow[dr] & \\
S_3 \arrow[ur] & & S_2 \arrow[ll, dotted] \arrow[ur] & & S_1 \arrow[ll, dotted]
\end{tikzcd}$$

Let $\mathcal{A}$ be the full additive subcategory generated by the indecomposables P_2, P_1, S_2, I_2. Then $\mathcal{A}$ is an intersection $\mathcal{F} \cap \mathcal{T}'$ where $\mathcal{F}$ is the torsion free class of the hereditary torsion pair $(\mathcal{T}, \mathcal{F}) = (\operatorname{add}(S_1), \operatorname{add}(S_3 \oplus P_2 \oplus P_1 \oplus S_2 \oplus I_2))$ and $\mathcal{T}'$ is the torsion class of the cohereditary torsion pair $(\mathcal{T}', \mathcal{F}') = (\operatorname{add}(P_2 \oplus P_2 \oplus S_2 \oplus I_2 \oplus S_1), \operatorname{add}(S_3))$ of $\operatorname{rep} Q$. Following [40, Theorem 3.2] and [36, Theorem 2], we conclude that $\mathcal{A}$ is an integral quasi-abelian category with exact structure $\mathcal{E}_{all}$ generated by the Auslander-Reiten sequence

$$0 \longrightarrow P_2 \longrightarrow P_1 \oplus S_2 \longrightarrow I_2 \longrightarrow 0$$

We verify that the axiom (AS) fails; to that end, we consider the following admissible monics in $\mathcal{A}$:

$$\begin{tikzcd}
 & P_2 \arrow[d, tail, "{\begin{bmatrix} a \\ b \end{bmatrix}}"] \\
P_1 \arrow[r, tail, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] & P_1 \oplus S_2
\end{tikzcd}$$

The pullback along these monics in the abelian category $\operatorname{rep} Q$ is given by the object S_3, but this is not available in $\mathcal{A}$. Being quasi-abelian and so pre-abelian, $\mathcal{A}$ admits a pullback which is a subobject of the abelian pullback, thus the zero object. Hence, we have in $\mathcal{A}$ that the intersection along the given monics is

$P_1 \cap P_2 = 0$, and therefore, by Lemma 4.7, $P_1 + P_2 = P_1 \oplus P_2$. However, the direct sum $P_1 \oplus P_2$ is not an admissible subobject of $P_1 \oplus S_2$, thus the axiom (AS) fails.

The next results show that having admissible intersections is not enough to be abelian. The computation of the previous example suggests to look for a quasi-abelian subcategory of an abelian category where the pullback diagrams in axiom (AI) coincide with the abelian ones (this is not in general true for all kernels), thus one gets an AI-subcategory. Typical examples of such quasi-abelian but non-abelian categories arise in functional analysis (see [24] for more examples):

Definition 4.14. We denote by **Ban** the category of Banach spaces (over the field of real numbers). The objects of **Ban** are the complete normed $\mathbb{R}$-vector spaces, and morphisms are continuous linear maps.

The kernel of a morphism $f : X \to Y$ in **Ban** is the linear kernel $f^{-1}(0) \rightarrowtail X$, however the cokernel

$$Y \twoheadrightarrow Y/\overline{f(X)}$$

in **Ban** is in general different from the linear cokernel $Y \to Y/f(X)$. Thus $f : X \to Y$ is an admissible monic in **Ban** precisely when f is a monomorphism such that $f(X)$ is closed in Y. The Open Mapping Theorem for Banach spaces guarantees that an admissible monic $f : X \to Y$ is an isomorphism onto $f(X)$. In fact the class $\mathcal{E} = \mathcal{E}_{all}$ of all kernel-cokernel pairs coincides with the class of short exact sequences of bounded linear maps, see [11, IV.2]. It is well-known that the category **Ban** is quasi-abelian with the maximal exact structure $\mathcal{E}_{all}$, but it is not abelian. We verify here the admissible intersection property and we reprove, using Theorem 4.12 that **Ban** is quasi-abelian:

Theorem 4.15. *The category **Ban** of Banach spaces, equipped with the maximal exact structure $\mathcal{E} = \mathcal{E}_{all}$, is an AI-category.*

Proof. Consider two $\mathcal{E}$-subobjects (X_0, f_0), (X_1, f_1) of an object X in **Ban**. Since the admissible monics f_i are isomorphisms onto their range $f_i(X_i)$, we can identify X_0 and X_1 with closed subspaces of X. The intersection of closed subspaces is closed, therefore we have the following diagram of closed embeddings (which are admissible monics):

$$\begin{tikzcd} X_0 \cap X_1 \arrow[r, tail, "i_1"] \arrow[d, tail, "i_0"'] & X_1 \arrow[d, tail, "f_1"] \\ X_0 \arrow[r, tail, "f_0"'] & X \end{tikzcd}$$

From [23], we know that the object $X_0 \cap X_1$ satisfies the pullback property from axiom (AI) in mod $\mathbb{R}$. Since the pullback can be written as kernel (Remark 4.5) and kernels in **Ban** are the kernels in mod $\mathbb{R}$, we conclude that the (AI)-axiom is satisfied: The pullback along admissible monics exists, and yields admissible monics. □

Remark 4.16. While **Ban** satisfies the admissible intersection property, it does not satisfy the admissible sum property and so it is not abelian by Theorem 3.7. Consider for a moment the second axiom (AS) in the setting studied in the proof of the previous theorem: It is shown in [5, Chapter 3.1] that *both,* the intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$ (as subvector spaces of X) admit norms turning them into Banach spaces, satisfying that

$$X_0 \cap X_1 \hookrightarrow X_i \hookrightarrow X_0 + X_1$$

are continuous embeddings for $i = 0, 1$. In fact, the whole interval between $X_0 \cap X_1$ and $X_0 + X_1$ is studied in [29], as *interpolations* between intersection and sum. We summarize the situation in the following diagram:

$$\begin{tikzcd}
X_0 \cap X_1 \arrow[r, tail, "i_1"] \arrow[d, tail, "i_0"'] & X_1 \arrow[d, tail, "j_1"] \arrow[ddr, tail, bend left, "f_1"] & \\
X_0 \arrow[r, tail, "j_0"'] \arrow[drr, tail, bend right, "f_0"'] & X_0 + X_1 \arrow[dr, "r"] & \\
& & X
\end{tikzcd}$$

The sum $X_0 + X_1$ is the pushout in $\mathrm{mod}\,\mathbb{R}$, hence satisfies the pushout property in **Ban** since the kernel-cokernel pairs of bounded maps in $\mathrm{mod}\,\mathbb{R}$ are also exact in **Ban**. However, the inclusion map $r : X_0+X_1 \to X$ (which is bounded, thus continuous) is not an admissible monic in general: The norm on $X_0 + X_1$ is given in [5, Chapter 3.1] by

$$\|x\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} \mid x_0 + x_1 = x\},$$

and with respect to this norm, the subspace X_0+X_1 in X is not necessarily closed. In fact, since we show, in Proposition 4.21, that any AIS-category is abelian, and **Ban** is not abelian, we conclude that *it is impossible to define a norm on the subspace $X_0 + X_1$ of X which turns $X_0 + X_1$ into a complete closed subspace of X.* Rephrased in terms of the poset $\mathcal{P}_X^{\mathcal{E}}$ of closed subspaces of a Banach space X, the observation above shows that the sum $X_0 + X_1$ is in general not an element of $\mathcal{P}_X^{\mathcal{E}}$. However, the intersection $X_0 \cap X_1$ is defined in $\mathcal{P}_X^{\mathcal{E}}$, turning the poset $\mathcal{P}_X^{\mathcal{E}}$ into a meet semi-lattice. It is well-known that a meet semi-lattice is a lattice when it is *complete*, i.e. closed under arbitrary intersections, and admits a unique maximal element (which is X here). However, this is also not true for the category **Ban** equipped with $\mathcal{E} = \mathcal{E}_{all}$ since an infinite intersection of closed subspaces need not be closed.

We proved in Theorem 4.12 that admitting admissible intersections is enough to be quasi-abelian, but it has been proved recently in [24, Theorem 6.1] that the converse also holds, and hence together a new characterisation of quasi-abelian categories is established. For the convenience of the reader, and with the kind permission of the authors of [24], we also include their part of the proof below:

Theorem 4.17. ***(Brüstle, Hassoun, Shah, Tattar, Wegner)*** *A category $(\mathcal{A}, \mathcal{E}_{max})$ is quasi-abelian if and only if it is an AI-category.*

Proof. ($\Longleftarrow$) By Theorem 4.12 every AI-category is quasi-abelian.

($\Longrightarrow$) Let $\mathcal{A}$ be a quasi-abelian category. Endowing it with the class $\mathcal{E}$ of all kernel-cokernel pairs in $\mathcal{A}$ yields an exact category $(\mathcal{A}, \mathcal{E})$ as $\mathcal{A}$ is quasi-abelian; see [39, Rmk. 1.1.11]. The class of admissible monomorphisms in $(\mathcal{A}, \mathcal{E})$ is thus precisely the class of kernels in $\mathcal{A}$. Let $c\colon B \rightarrowtail D$ and $d\colon C \rightarrowtail D$ be arbitrary admissible monomorphisms in $(\mathcal{A}, \mathcal{E})$, i.e. c, d are kernels. Then in the pullback diagram

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, tail, "c"] \\
C \arrow[r, tail, "d"'] & D
\end{tikzcd}$$

the morphisms a and b are also kernels in $\mathcal{A}$ by the dual of Kelly [26, Prop. 5.2]. That is, a, b are admissible monomorphisms, and we see that $(\mathcal{A}, \mathcal{E})$ has admissible intersections. $\square$

4.3. AIS-categories and abelian categories

In this subsection we prove that the categories satisfying both the (AI) and the (AS) axioms are exactly the abelian categories.

Proposition 4.18. *Let X be an object in an AIS-category $(\mathcal{A}, \mathcal{E})$. Then the poset $\mathcal{P}_X^{\mathcal{E}}$ forms a lattice under the operations*

$$(\mathcal{P}_X^{\mathcal{E}}, \leq, \cap_X, +_X)$$

where $\cap_X$ and $+_X$ are the intersection and sum operations defined in Definition 4.4.

Proof. We need to verify the axioms of Definition 2.14. The first and second axioms follow directly from Remark 4.5. For the third axiom, we have to show

$$Y + (Y \cap Z) = Y = Y \cap (Y + Z)$$

for $\mathcal{E}$-subobjects Y, Z of X. We give the proof of the first equality here, the second one being similar: By axiom (AI), we know that there is an admissible monic $Y \cap Z \rightarrowtail Y$. Remark 4.6 applied to this inclusion and the object Y yields

$$(Y \cap Z) + Y \rightarrowtail Y + Y.$$

Since $Y + Y = Y$ by Remark 4.6, we have an admissible monic $Y + (Y \cap Z) \rightarrowtail Y$. But by the axiom (AS), there is also an admissible monic from Y into the sum of Y with $Y \cap Z$, therefore we have

$$Y \rightarrowtail (Y \cap Z) + Y \rightarrowtail Y.$$

This shows that the monics are isomorphisms, hence equalities in the poset $\mathcal{P}_X^{\mathcal{E}}$. $\square$

Lemma 4.19. *[23, Corollary 4.11] Let $\mathcal{A}$ be an abelian category. Then $(\mathcal{A}, \mathcal{E}_{all})$ is an AIS-category.*

Lemma 4.20. *Let $\mathcal{A}$ be a quasi-abelian category and $\mathcal{E} = \mathcal{E}_{all}$. Suppose that every monomorphism in $\mathcal{A}$ is a kernel, then $\mathcal{A}$ is abelian. Dually, if every epimorphism is a cokernel, then $\mathcal{A}$ is abelian.*

Proof. Let $f : X \to Y$ be an arbitrary morphism in $\mathcal{A}$ we will show that f is admissible, whence it follows that $\mathcal{A}$ is abelian by Lemma 3.4. Recall that there is a commutative diagram in $\mathcal{A}$

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[d, tail] & \operatorname{Coker} f \\
X \arrow[r, "f"] \arrow[d, two heads, "c"] & Y \arrow[u, two heads] \\
\operatorname{Coim} f \arrow[r, "\bar{f}"] & \operatorname{Im} f \arrow[u, tail, "i"]
\end{tikzcd}$$

where $\bar{f}$ is both monic and epic (see [36, Section 1] for details) and the columns are $\mathcal{E}$-sequences since $\mathcal{A}$ is quasi-abelian and $\mathcal{E} = \mathcal{E}_{all}$. By assumption, the composition $i\bar{f}$ is a kernel and therefore an admissible monic since $\mathcal{A}$ is quasi-abelian. Thus the decomposition $f = (i\bar{f})c$ shows that f is admissible. The proof of the dual statement is similar. $\square$

Proposition 4.21. *Let $(\mathcal{A}, \mathcal{E})$ be an AIS-category. Then $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{all}$.*

Proof. By Theorem 4.12, $\mathcal{A}$ is quasi-abelian and $\mathcal{E} = \mathcal{E}_{all}$. Thus, by Lemma 4.20, it is enough to show that every monomorphism $f : X \to Y$ in $\mathcal{A}$ is a kernel. To this end, consider the $\mathcal{E}$-subobjects given by two sections

$$\begin{bmatrix} 1 \\ f \end{bmatrix} : X \rightarrowtail X \oplus Y$$
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} : X \rightarrowtail X \oplus Y.$$

By computation their intersection is the zero-object

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner" near end] & X \arrow[d, tail, "{\begin{bmatrix} 1 \\ f \end{bmatrix}}"] \\ X \arrow[r, tail, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] & X \oplus Y. \end{tikzcd}$$

Thus, by Lemma 4.7, their sum is given by the direct sum $X \oplus X$

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner" near start] & X \arrow[d, tail, "{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}"] \arrow[ddr, tail, bend left, "{\begin{bmatrix} 1 \\ f \end{bmatrix}}"] & \\ X \arrow[r, tail, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] \arrow[drr, tail, bend right, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] & X \oplus X \arrow[dr, tail, "u"] & \\ & & X \oplus Y \end{tikzcd}$$

where $u = \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix}$ is an admissible monic since $(\mathcal{A}, \mathcal{E})$ is (AIS). Now, by [10, Corollary 2.18], f is an admissible monic and we are done. $\square$

Theorem 4.22. *An exact category $(\mathcal{A}, \mathcal{E})$ is an AIS-category if and only if $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{all}$.*

Proof. By using Lemma 4.19 and Proposition 4.21. $\square$

Now that we know that the AIS-categories are precisely the abelian ones with their maximal exact structure, let us recall the second isomorphism theorem for an abelian category with its maximal exact structure, but using the language of exact categories where the intersection and sum defined in Definition 4.4 by pullbacks and pushouts are always admissible. This result will be useful later to prove that the abelian categories are diamond exact categories in the sense of Definition 5.7 (we refer the reader to [23, Lemma 5.2] for the proof):

Proposition 4.23. ***(The second $\mathcal{E}$-isomorphism theorem)*** *Assume that $(\mathcal{A}, \mathcal{E})$ is an AIS-exact category. Let X and Y be $\mathcal{E}$-subobjects of an object Z in $\mathcal{A}$. Then there is the following short exact sequence:*

$$X \cap_Z Y \rightarrowtail Y \twoheadrightarrow (X +_Z Y)/X.$$

In other words, there is an isomorphism (parallelogram identity):

$$Y/(X \cap_Z Y) \cong (X +_Z Y)/X.$$

5. The diamond exact categories

In this section we address the drawbacks of the intersection and sum in the previous section by introducing a general notion of intersection and sum that applies to exact categories. We then use this to introduce a class of exact categories - the diamond exact categories - and show that these satisfy the $\mathcal{E}$-Jordan-Hölder property as in Definition 5.1.

5.1. Jordan-Hölder property

Definition 5.1. Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A finite $\mathcal{E}$-composition series for an object X of $\mathcal{A}$ is a sequence

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{n-2}}{\rightarrowtail} X_{n-1} \overset{i_{n-1}}{\rightarrowtail} X_n = X \tag{2}$$

where all i_l are *proper admissible monics* with $\mathcal{E}$-simple cokernel. We say an exact category $(\mathcal{A}, \mathcal{E})$ has the *($\mathcal{E}-$)Jordan-Hölder property* or is a *Jordan-Hölder exact category* if any two finite $\mathcal{E}$-composition series for an object X of $\mathcal{A}$

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{m-2}}{\rightarrowtail} X_{m-1} \overset{i_{m-1}}{\rightarrowtail} X_m = X$$

and

$$0 = X'_0 \overset{i'_0}{\rightarrowtail} X'_1 \overset{i'_1}{\rightarrowtail} \dots \overset{i'_{n-2}}{\rightarrowtail} X'_{n-1} \overset{i'_{n-1}}{\rightarrowtail} X'_n = X$$

are equivalent, that is, they have the same length and the same composition factors, up to permutation and isomorphism.

Remark 5.2. As shown in [23], one can use the same steps as in [4] and the $\mathcal{E}$-Schur lemma to prove that every AIS-category $(\mathcal{A}, \mathcal{E})$ is a Jordan-Hölder exact category.

5.2. General intersection and sum

For an AIS-category $(\mathcal{A}, \mathcal{E})$, or equivalently, for an abelian category $\mathcal{A}$ with maximal exact structure $\mathcal{E}_{all}$, the intersection of two subobjects of X is defined as the pullback of their monomorphisms in X and their sum is defined as the pushout of this pullback, which is also admissible. In terms of the poset $\mathcal{P}_X^{\mathcal{E}}$ of $\mathcal{E}$-subobjects of X, this means that $\mathcal{P}_X^{\mathcal{E}}$ forms a lattice as shown in Proposition 4.18. However, in general the poset $\mathcal{P}_X^{\mathcal{E}}$ is not a lattice, even when the $\mathcal{E}$-Jordan-Hölder property holds for the exact category $(\mathcal{A}, \mathcal{E})$, as the following simple examples demonstrate.

Example 5.3. Let $\mathcal{A}$ be the category of all even dimension k-vector spaces endowed with the split exact structure $\mathcal{E} = \mathcal{E}_{min}$. Then the $\mathcal{E}$-simple objects are precisely the two-dimensional vector spaces, and the Jordan-Hölder property is clearly satisfied. Consider the object $X = k^6$ with basis $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the two elements of $\mathcal{P}_X^{\mathcal{E}}$ given by

$$V_1 =< v_1, v_2, v_3, v_4 > \quad \text{and} \quad V_2 =< v_2, v_3, v_4, v_5 > .$$

The intersection $V_1 \cap V_2$ in $\operatorname{mod} k$ is $V_3 =< v_2, v_3, v_4 >$. But since V_3 is not in $\mathcal{A}$, every two-dimensional subspace U of V_3 is a maximal lower bound for both V_1 and V_2, when we view (U, f) as an element in $\mathcal{P}_X^{\mathcal{E}}$

with its inclusion map f. Therefore $\mathcal{P}_X^{\mathcal{E}}$ is not a lattice, and the intersection of V_1 and V_2 is not unique in $(\mathcal{A}, \mathcal{E})$, in fact it is an infinite set formed by all embeddings (U, f) of maximal proper subspaces U of V_3.

Example 5.4. A similar phenomenon can be observed studying the additive category $\mathcal{A} = \operatorname{rep} A_2$ of representations of the quiver of type A_2, endowed with the minimal exact structure $\mathcal{E} = \mathcal{E}_{min}$. We denote the simple representations by S_1 and S_2, and the indecomposable projective-injective representation by P_1. Then there is a non-split indecomposable short exact sequence in $\mathcal{A}$

$$0 \longrightarrow S_2 \xrightarrow{f} P_1 \xrightarrow{g} S_1 \longrightarrow 0$$

which is not admissible in $\mathcal{E}_{min}$. Therefore $(\mathcal{A}, \mathcal{E}_{min})$ is not an AI-category by Proposition 4.8. Choosing the object $X = S_2 \oplus P_1 \oplus S_1$, we observe that there are many maximal $\mathcal{E}$-subobjects of X with quotient S_1 given by $(S_2 \oplus P_1, \alpha_\lambda)$ with $\lambda \in k$, where

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \lambda g \end{bmatrix} = \alpha_\lambda : S_2 \oplus P_1 \to X = S_2 \oplus P_1 \oplus S_1.$$

Each of these admit many maximal $\mathcal{E}$-subobjects with quotient P_1 given by $(S_2 \oplus P_1, \beta_\mu)$ with $\mu \in k$ where

$$\begin{bmatrix} 1 \\ \mu f \end{bmatrix} = \beta_\mu : S_2 \to S_2 \oplus P_1.$$

The preceding examples motivate the following definition, where we allow the (generalised) intersection and sum to be a set of objects:

Definition 5.5. Let (A_i, f_i), $i \in I$, be a collection of $\mathcal{E}$-subobjects of X indexed by a set I. We denote the set of all their common admissible subobjects with respect to X as

$$\operatorname{Sub}_X(\{(A_i, f_i) \mid i \in I\}) := \{(Y, h) \in P_X^{\mathcal{E}} \mid (Y, h_i) \in \mathcal{P}_A^{\mathcal{E}},\ h = f_i h_i\ \forall i \in I\}$$

and define the $\mathcal{E}$-relative intersection of the (A_i, f_i) in $\mathcal{P}_X^{\mathcal{E}}$ as

$$\operatorname{Int}_X(\{(A_i, f_i) \mid i \in I\}) := \operatorname{Max}(\operatorname{Sub}_X(\{(A_i, f_i) \mid i \in I\}),$$

the set of maximal elements in $\operatorname{Sub}_X(\{(A_i, f_i) \mid i \in I\})$ (where we define the generalised intersection over the empty set to be $\{0\}$). Dually, we denote the set of all common superobjects of A and B as

$$\operatorname{Sup}_X(\{(A_i, f_i) \mid i \in I\}) := \{\ (Y, h) \in \mathcal{P}_X^{\mathcal{E}} \mid (A_i, g_i) \in \mathcal{P}_Y^{\mathcal{E}},\ f_i = h g_i\ \forall i \in I\}$$

and define the $\mathcal{E}$-relative sum of A and B in $\mathcal{P}_X^{\mathcal{E}}$ as

$$\operatorname{Sum}_X(\{(A_i, f_i) \mid i \in I\}) := \operatorname{Min}(\operatorname{Sup}_X(\{(A_i, f_i) \mid i \in I\})),$$

the set of minimal elements in $\operatorname{Sup}_X(\{(A_i, f_i) \mid i \in I\})$.

Example 5.6. In the setup of Example 5.3, the objects V_1 and V_2 have as $\mathcal{E}$-relative intersection in $\mathcal{P}_X^{\mathcal{E}}$ the Grassmannian $\operatorname{Int}_X(V_1, V_2) = Gr(2, 3)$ of all maximal proper subspaces of V_3. The set $\operatorname{Sum}_X(V_1, V_2)$ however consists only of the element X itself. In Example 5.4, any two of the objects $(S_2 \oplus P_1, \alpha_\lambda)$ have an infinite intersection containing all elements (S_2, β_μ) of $\mathcal{P}_X^{\mathcal{E}}$, and conversely, any two of the (S_2, β_μ) have an infinite sum containing all the objects $(S_2 \oplus P_1, \alpha_\lambda)$.

5.3. The diamond categories are Jordan-Hölder exact categories

In this section we prove the $\mathcal{E}$-Jordan-Hölder property in a more general context than abelian categories, namely for exact categories that we call *the diamond exact categories*:

Definition 5.7. (**Diamond Axiom**) Let (A, f) and (B, g) be two distinct maximal $\mathcal{E}$-sub-objects in $\mathcal{P}_X$, that is, their cokernels X/A and X/B are $\mathcal{E}$-simple. We say that (A, f) and (B, g) satisfy the *diamond axiom* if for every $Y \in \mathrm{Int}_X(A, B)$ we have that A/Y and B/Y are both $\mathcal{E}$-simple and the elements of the sets $\{X/A, A/Y\}\{X/B, B/Y\}$ are equal up to permutation and isomorphism.

$$\begin{tikzcd} & A \arrow[dr, tail, "f"] & \\ Y \arrow[ur, tail] \arrow[dr, tail] & & X \\ & B \arrow[ur, tail, "g"'] & \end{tikzcd}$$

A *diamond* exact category $(\mathcal{A}, \mathcal{E})$ is an exact category that satisfies the diamond axiom for each pair of maximal subobjects A and B of some object X.

Remark 5.8. When $\mathcal{A}$ is an abelian category, then for each object X we have that $\mathrm{Int}_X(A, B)$ and $\mathrm{Sum}_X(A, B)$ are given by the unique objects $A \cap_X B$ and $A +_X B$, respectively. Lemma 4.23 then ensures that the diamond axiom is always satisfied. We conclude that every abelian category is a diamond exact category.

Remark 5.9. Note that Lemma 4.23 guarantees that we always have crosswise isomorphisms

$$X/A \cong B/Y \quad \text{and} \quad X/B \cong A/Y$$

in the abelian case. However, Example 5.4 shows that one can have the lengthwise isomorphisms

$$X/A \cong A/Y \quad \text{and} \quad X/B \cong B/Y$$

when the poset $\mathcal{P}_X^{\mathcal{E}}$ is not a lattice.

Lemma 5.10. *Assume that an object X in a diamond exact category $\mathcal{A}$ has a composition series of length n*

$$0 = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_n = X.$$

If (C, f) is a maximal element in $\mathcal{P}_X$, then there exists a composition series of X through C of length n:

$$0 = C_0 \rightarrowtail C_1 \rightarrowtail \dots \rightarrowtail C_{n-2} \rightarrowtail C \overset{f}{\rightarrowtail} X.$$

Proof. By induction on n. For $n = 1$, this is obvious because $C = 0$. Assume now $n \geq 2$. If $B_{n-1} = C$ as elements in $\mathcal{P}_X^{\mathcal{E}}$, we can use the given composition series of X. Otherwise, consider an element $Y \in \mathrm{Int}_X(B_{n-1}, C)$:

$$\begin{tikzcd}
0 = B_0 \arrow[r, tail] & \dots \arrow[r, tail] & B_{n-1} \arrow[dr, two heads] & \\
& Y \arrow[ur, tail] \arrow[dr, tail] & & X \\
& & C \arrow[ur, two heads] &
\end{tikzcd}$$

By the diamond axiom, both quotients B_{n-1}/Y and C/Y are $\mathcal{E}$-simple since B_{n-1} and C are maximal elements in $\mathcal{P}_X^{\mathcal{E}}$. Thus we have a composition series of length $n-1$

$$0 = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_{n-1} = X'$$

and Y is maximal in $\mathcal{P}_{X'}^{\mathcal{E}}$. Induction hypothesis implies that there exists a composition series of X' through Y of length $n-1$. Replacing $Y \rightarrowtail B_{n-1}$ in this series by $Y \rightarrowtail C \rightarrowtail X$ yields a composition series of X through C of length n:

$$0 = Y_0 \rightarrowtail \dots \rightarrowtail Y_{n-3} \rightarrowtail Y \rightarrowtail C \xrightarrow{f} X. \qquad \square$$

Theorem 5.11. *Every diamond exact category is a Jordan-Hölder exact category.*

Proof. Following the strategy of the proof in [30, Chapter 4.5], assume we are given two composition series

$$0 = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_n = X$$

and

$$0 = C_0 \rightarrowtail C_1 \rightarrowtail \dots \rightarrowtail C_m = X.$$

We proceed by induction on n. For $n = 1$, the object X is $\mathcal{E}$-simple and the statement clearly holds. Assume now $n \geq 2$. For any object $Y \in \operatorname{Int}_X(B_{n-1}, C_{m-1})$ we obtain the following diagram:

$$\begin{tikzcd}
0 = B_0 \arrow[r, tail] & \dots \arrow[r, tail] & B_{n-1} \arrow[dr, two heads] & \\
& Y \arrow[ur, tail] \arrow[dr, tail] & & X \\
0 = C_0 \arrow[r, tail] & \dots \arrow[r, tail] & C_{m-1} \arrow[ur, two heads] &
\end{tikzcd}$$

The diamond axiom applied to the maximal $\mathcal{E}$-subobjects B_{n-1}, C_{m-1} of X yields that Y is maximal in both B_{n-1} and C_{m-1}. Lemma 5.10 applied to the maximal element Y of B_{n-1} yields a composition series

$$0 = Y_0 \rightarrowtail \dots \rightarrowtail Y_{n-3} \rightarrowtail Y \rightarrowtail B_{n-1}$$

of length $n-1$. Moreover, Lemma 5.10 applied to the maximal element Y of C_{m-1} yields a composition series

$$0 = Y'_0 \rightarrowtail \dots \rightarrowtail Y'_{m-3} \rightarrowtail Y \rightarrowtail C_{m-1}$$

of length $m-1$. This gives two composition series of the object Y of length $n-2$ and $m-2$, respectively. By induction hypothesis, we conclude that $n-2 = m-2$ (thus $n = m$), and that these two composition

series of Y have the same composition factors, up to permutation and isomorphism.
Consider now the following diagram:

$$\begin{tikzcd}
0 = B_0 \arrow[r, tail] & \dots \arrow[r, tail] & B_{n-2} \arrow[r, tail] & B_{n-1} \arrow[dr, tail] & \\
0 \arrow[r, tail] & \dots \arrow[r, tail] & Y \arrow[ur, tail] \arrow[dr, tail] & & X \\
0 = C_0 \arrow[r, tail] & \dots \arrow[r, tail] & C_{n-2} \arrow[r, tail] & C_{n-1} \arrow[ur, tail] &
\end{tikzcd}$$

By induction hypothesis, the two composition series

$$0 = B_0 \rightarrowtail \dots \rightarrowtail B_{n-2} \rightarrowtail B_{n-1}$$

and

$$0 = Y_0 \rightarrowtail \dots \rightarrowtail Y \rightarrowtail B_{n-1}$$

are equivalent.
In the same way, the two composition series

$$0 = C_0 \rightarrowtail \dots \rightarrowtail C_{n-2} \rightarrowtail C_{n-1}$$

and

$$0 = Y_0' \rightarrowtail \dots \rightarrowtail Y \rightarrowtail C_{n-1}$$

are equivalent.

By the diamond axiom, the sets of quotients $\{X/B_{n-1}, B_{n-1}/Y\}$ and $\{X/C_{n-1}, C_{n-1}/Y\}$ are equal, up to isomorphism. Using this fact and comparing the four composition series of length $n-1$ above, we conclude that the two composition series given in the beginning have the same composition factors up to permutations and isomorphism. $\square$

We provide in Section 6 examples of diamond categories that are not abelian categories with $\mathcal{E}_{all}$. However, not every exact category $(\mathcal{A}, \mathcal{E})$ is diamond, or Jordan-Hölder, even if $\mathcal{A}$ is abelian, as the following example demonstrates.

Example 5.12. Consider the category $\mathcal{A} = \operatorname{rep} Q$ of representations of the quiver

$$\begin{tikzcd}
& 1 \arrow[dl] \arrow[dr] & \\
2 & & 3
\end{tikzcd}$$

The Auslander-Reiten quiver of $\mathcal{A}$ is as follows:

$$\begin{tikzcd}
S_2 \arrow[dr] & & I_3 \arrow[ll, dashed] \arrow[dr] & \\
& P_1 \arrow[ur] \arrow[dr] & & I_1 = S_1 \arrow[ll, dashed] \\
S_3 \arrow[ur] & & I_2 \arrow[ll, dashed] \arrow[ur] &
\end{tikzcd}$$

By Theorem 2.13, each exact structure on $\mathcal{A}$ is uniquely determined by the set of Auslander-Reiten sequences which it contains. Consider the exact structure $\mathcal{E}$ containing the Auslander-Reiten sequences

$$0 \longrightarrow S_2 \longrightarrow P_1 \longrightarrow I_3 \longrightarrow 0 \tag{AR1}$$

$$0 \longrightarrow S_3 \longrightarrow P_1 \longrightarrow I_2 \longrightarrow 0 \tag{AR2}$$

Then $(\mathcal{A}, \mathcal{E})$ is not Jordan-Hölder, and it is also not a diamond category. Indeed, we have that the simples S_2 and S_3 are maximal subobjects of P_1, but the quotient sets $\{S_2, P_1/S_2 = I_3\}$ and $\{S_3, P_1/S_3 = I_2\}$ are not isomorphic.

6. $\mathcal{E}$-Artin-Wedderburn categories

We use the notion of generalised intersection to define a version of the Jacobson radical relative to an exact structure $\mathcal{E}$. This allows us to show the Jordan-Hölder property for Krull-Schmidt categories under the assumption that this $\mathcal{E}$-radical behaves well with respect to direct sums of $\mathcal{E}$-simple objects, that is, the exact structure satisfies an exact analog of the Artin-Wedderburn theorem. We then classify all such exact structures in $\operatorname{rep}\Lambda$ where Λ is a Nakayama algebra and furthermore note that these are all Jordan-Hölder exact structures on $\operatorname{rep}\Lambda$.

Throughout this section, we assume all categories to be Krull-Schmidt categories. Recall that a is a Krull-Schmidt category an additive category, $\mathcal{A}$, such that each object decomposes into a finite direct sum of indecomposable objects having local endomorphism rings and that this decomposition is unique up to isomorphism and permutation of summands. In particular, in this case $(\mathcal{A}, \mathcal{E}_{min})$ is a Jordan-Hölder category.

6.1. $\mathcal{E}$-Jacobson radical

Let $(\mathcal{A}, \mathcal{E})$ be an essentially small Krull-Schmidt exact category. We introduce a Jacobson radical for exact categories.

Definition 6.1. Let $X \in \mathcal{A}$, we define the *$\mathcal{E}$-Jacobson radical* to be the generalised intersection

$$\operatorname{rad}_{\mathcal{E}}(X) := \operatorname{Int}_X\{(Y, f) \in \mathcal{S}_X \mid (Y, f) \in \operatorname{Max}(\mathcal{S}_X)\}.$$

Note that, by Definition 5.5, $\operatorname{rad}_{\mathcal{E}} S = \{0\}$ for all $\mathcal{E}$-simple objects S.

Proposition 6.2. *Consider $X, Y \in \mathcal{A}$ and $r : R \rightarrowtail X$.*

a) *For all $(R, r) \in rad_{\mathcal{E}}(X)$, $rad_{\mathcal{E}}(Coker(r)) = \{0\}$.*
b) *For all $(Z, g) \in \mathcal{S}_X$, Z is an $\mathcal{E}$-subobject of some $(R, r) \in rad_{\mathcal{E}}(X)$ if and only if $pg = 0$ for all $\mathcal{E}$-simple quotients $p : X \twoheadrightarrow S$ of X.*

Proof. a) Let $(R, r) \in \operatorname{rad}_{\mathcal{E}}(X)$ and $(Q, q) \in \operatorname{rad}_{\mathcal{E}}(X/R)$ corresponding to $Q' \rightarrowtail X$ via the Fourth $\mathcal{E}$-isomorphism Theorem (Proposition 3.8). By same result and since $(R, r) \in \operatorname{rad}_{\mathcal{E}}(X)$ we have that the maximal $\mathcal{E}$-subobjects of X correspond exactly to maximal $\mathcal{E}$-subobjects of X/R. Hence, as Q is an $\mathcal{E}$-subobject of every $\mathcal{E}$-maximal subobject of X/R, we have that Q' is an $\mathcal{E}$-subobject of every maximal $\mathcal{E}$-subobject of X. Thus, by definition of the generalised intersection, since $R \rightarrowtail Q'$ we deduce that $R \cong Q'$ so $Q \cong Q'/R \cong 0$.

b) The claim follows from the observation that admissible epimorphisms $X \twoheadrightarrow S$ with S being $\mathcal{E}$-simple correspond exactly to maximal $\mathcal{E}$-subobjects of X. $\square$

Definition 6.3. An object $X \in \mathcal{A}$ is called $\mathcal{E}$*-semisimple* if it can be written as a finite direct sum of $\mathcal{E}$-simple objects.

We study exact categories where the $\mathcal{E}$-semisimple objects have nice characterisations:

Definition 6.4. An exact structure $\mathcal{E}$ on $\mathcal{A}$ is called *Artin-Wedderburn* if for any object $X \in \mathcal{A}$ the following properties are equivalent:

(AW1) Every sequence in $\mathcal{E}$ of the form $A \rightarrowtail X \twoheadrightarrow X/A$ splits,
(AW2) X is $\mathcal{E}$-semisimple,
(AW3) $\operatorname{rad}_{\mathcal{E}}(X) = \{0\}$.

We say in this case that $(\mathcal{A}, \mathcal{E})$ is an $\mathcal{E}$*-Artin-Wedderburn category.*

Remark 6.5.

a) The implication (AW1) $\Rightarrow$ (AW2) always holds for Krull-Schmidt categories. Indeed, suppose X is not $\mathcal{E}$-semisimple. Then in the decomposition of X as a direct sum of indecomposables, $X \cong \bigoplus_{i=1}^{n} X_i$, there exists $1 \leq i \leq n$ such that X_i is not $\mathcal{E}$-simple. Thus there exists a non-split $\mathcal{E}$-inflation $f : Y \rightarrowtail X$ and observe that composing f with the canonical inclusion $X_i \rightarrowtail X$ results in a non-split $\mathcal{E}$-inflation $Y \rightarrowtail X$.
We note that without the Krull-Schmidt assumption on our categories, this implication in general does not hold, even in the abelian case. A class of counterexamples is given by the continuous spectral categories. These are Grothendieck categories where every short exact sequence splits but there are no simple objects as every object is decomposable, see [33, Example 2.9] for examples of such categories.

b) The implication (AW2) $\Rightarrow$ (AW3) also always holds. Indeed, let S_i, $1 \leq i \leq n$ be $\mathcal{E}$-simple objects and $X = \bigoplus_{i=1}^{n} S_i$. Then observe that for all $1 \leq j \leq n$ that $\bigoplus_{i=1, i \neq j}^{n} S_i$ equipped with the canonical inclusion $f_j : \bigoplus_{i \neq j} S_i \rightarrowtail X$ is an $\mathcal{E}$-maximal subobject of X. Thus for every $(r : R \rightarrowtail X) \in \operatorname{rad}_{\mathcal{E}}(X)$, r factors through f_j for all $1 \leq j \leq n$ and we deduce that $r = 0$.

Example 6.6. Consider the category $\mathcal{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q: \qquad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

We classify which exact structures $\mathcal{E}$ on $\mathcal{A}$ are Artin-Wedderburn, and when $(\mathcal{A}, \mathcal{E})$ is a diamond or Jordan-Hölder category. The Auslander-Reiten quiver of $\mathcal{A}$ is

$$\begin{tikzcd}
 & P_3 \arrow[dr] & & S_1 \arrow[ll, dotted] \\
S_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ll, dotted] \arrow[ur] \arrow[dr] & \\
 & P_1 \arrow[ur] & & S_3 \arrow[ll, dotted]
\end{tikzcd}$$

and the Auslander Reiten sequences in $\mathcal{A}$ are

(1) $S_2 \to P_1 \oplus P_3 \to I_2$
(2) $P_3 \to I_2 \to S_1$

(3) $P_1 \to I_2 \to S_3$.

This example has been studied in [8, Example 4.2], and $\mathcal{A}$ admits precisely $2^3 = 8$ exact structures $\mathcal{E}$ corresponding to choosing some subset $\mathcal{B}$ of the three Auslander-Reiten sequences in $\mathcal{A}$, as discussed in Theorem 2.13. We denote the different exact structures accordingly as $\mathcal{E}_{min}, \mathcal{E}(1), \mathcal{E}(2), \mathcal{E}(3), \mathcal{E}(1,2), \mathcal{E}(1,3), \mathcal{E}(2,3), \mathcal{E}_{max}$, indicating the Auslander-Reiten sequences that are included.

Consider first the exact structure $\mathcal{E}(1)$ generated by the Auslander-Reiten sequence (1). Then the only non-split indecomposable $\mathcal{E}(1)$-sequence is (1) thus $P_1 \oplus P_3$ is $\mathcal{E}(1)$-semisimple and $(\mathcal{A}, \mathcal{E}(1))$ does not satisfy the implication (AW2) $\Rightarrow$ (AW1). The same object $P_1 \oplus P_3$ also shows that $(\mathcal{A}, \mathcal{E}(1))$ is not Jordan-Hölder (and hence not diamond) since there are non-equivalent $\mathcal{E}(1)$-composition series $0 \to S_2 \to P_1 \oplus P_3$ and $0 \to P_1 \to P_1 \oplus P_3$.

Now consider the exact structure $\mathcal{E}(2,3)$ on $\mathcal{A}$ generated by the sequences (2) and (3). As in Example 5.12 one can see that $(\mathcal{A}, \mathcal{E}(2,3))$ is not Jordan-Hölder nor diamond. Moreover, $\mathrm{rad}_{\mathcal{E}(2,3)}(I_2) = \{0\}$ but I_2 is not $\mathcal{E}(2,3)$-semisimple thus $(\mathcal{A}, \mathcal{E}(2,3))$ satisfies neither the implication (AW3) $\Rightarrow$ (AW1) nor (AW3) $\Rightarrow$ (AW2).

One may verify that all other exact structures $\mathcal{E}$ on $\mathcal{A}$ are Artin-Wedderburn, and also satisfy the diamond and Jordan-Hölder property, but only $(\mathcal{A}, \mathcal{E}_{max})$ is an AIS-category. We conclude that six of the eight exact structures are Jordan-Hölder, and in this example, the conditions being $\mathcal{E}$-Artin-Wedderburn, diamond and Jordan-Hölder are equivalent.

A further example of $\mathcal{E}$-Artin-Wedderburn categories is provided by the split exact structure:

Lemma 6.7. *$\mathcal{A}$ is an $\mathcal{E}_{min}$-Artin-Wedderburn category.*

Proof. For the exact structure $\mathcal{E} = \mathcal{E}_{min}$, we have that the admissible monics are precisely the sections, and the $\mathcal{E}$-simple objects are the indecomposables. Every object in $\mathcal{A}$ is thus $\mathcal{E}$-semisimple, and we clearly have the equivalence of (AW1) and (AW2). Since every X is $\mathcal{E}$-semisimple, the implication (AW3) $\Longrightarrow$ (AW2) is always true. □

As we have noted, for Krull-Schmidt categories, $(\mathcal{A}, \mathcal{E}_{min})$ is a Jordan-Hölder category. The following result further studies the relationship between Krull-Schmidt categories and the Jordan-Hölder property.

Theorem 6.8. *Let $(\mathcal{A}, \mathcal{E})$ be an $\mathcal{E}$-Artin-Wedderburn category. Then $(\mathcal{A}, \mathcal{E})$ is a Jordan-Hölder exact category.*

Proof. We show that $(\mathcal{A}, \mathcal{E})$ satisfies the Diamond Axiom 5.7. For that purpose, let

$$\begin{tikzcd} & A \arrow[dr, tail] & \\ C \arrow[ur, tail] \arrow[dr, tail] & & D \\ & B \arrow[ur, tail] & \end{tikzcd}$$

be a commutative diagram in $(\mathcal{A}, \mathcal{E})$ with D/A and D/B being $\mathcal{E}$-simple and $C \in \mathrm{Int}_D(A, B)$. By the Fourth $\mathcal{E}$-Isomorphism Theorem (Proposition 3.8), there is a commutative diagram

$$\begin{tikzcd} & A/C \arrow[dr, tail] & \\ 0 \arrow[ur, tail] \arrow[dr, tail] & & D/C \\ & B/C \arrow[ur, tail] & \end{tikzcd}$$

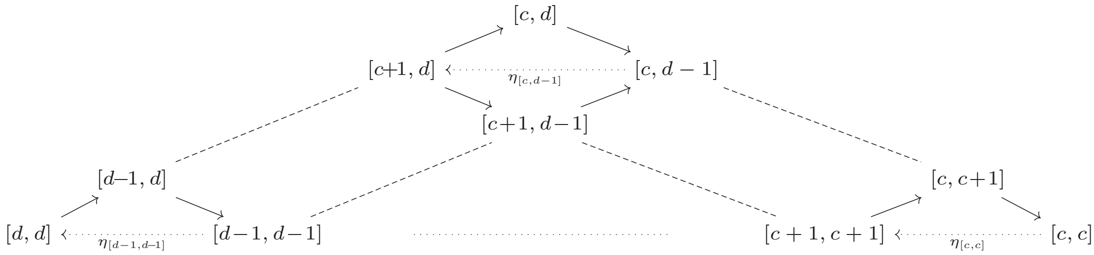

Fig. 1. Part of the Auslander-Reiten quiver of rep Λ containing the module $[c, d]$ and all of its simple composition factors.

with $(D/C)/(A/C) \cong D/A$ and $(D/C)/(B/C) \cong D/B$ being $\mathcal{E}$-simple and $\mathrm{Int}_{D/C}(A/C, B/C) = \{0\}$. Thus, it is enough to consider diagrams of the form

$$\begin{tikzcd} & X_0 \arrow[dr, tail, "f_0"] & \\ 0 \arrow[ur, tail] \arrow[dr, tail] & & Y \\ & X_1 \arrow[ur, tail, "f_1"'] & \end{tikzcd}$$

with Y/X_i being $\mathcal{E}$-simple for $i = 0, 1$ and $\mathrm{Int}_Y(X_0, X_1) = \{0\}$.

We must show that the X_i are $\mathcal{E}$-simple and that the sets $\{X_0, Y/X_0\}, \{X_1, Y/X_1\}$ are equal up to permutation and isomorphism of their elements.

If (X_0, f_1) and (X_1, f_1) are isomorphic as $\mathcal{E}$-subobjects of Y it follows that $X_0 \cong X_1$ is $\mathcal{E}$-simple since $\mathrm{Int}_Y(X_0, X_1) = \{0\}$. So we may assume that this is not the case. Observe that the (X_i, f_i) are both maximal $\mathcal{E}$-subobjects of Y. It follows that $\mathrm{rad}_{\mathcal{E}}(Y) \subset \mathrm{Int}_Y((X_0, f_0), (X_1, f_1)) = \{0\}$. Since $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artin-Wedderburn, the short exact sequences $X_i \overset{f_i}{\rightarrowtail} Y \twoheadrightarrow Y/X_i$ both split and Y is $\mathcal{E}$-semisimple. Thus $X_0 \oplus Y/X_0 \cong Y \cong X_1 \oplus Y/X_1$ and $X_0 \cong \bigoplus_{j=0}^{n} S_j$ and $X_1 \cong \bigoplus_{j=0}^{m} T_j$ with the S_j and T_j being $\mathcal{E}$-simple. As $(\mathcal{A}, \mathcal{E})$ is Krull-Schmidt, $n = m$ and the sets $\{S_0, \dots, S_n, Y/X_0\}, \{T_0, \dots, T_n, Y/X_1\}$ consist of the same objects, up to permutation and isomorphism. Without loss of generality, we may suppose that $S_0 \cong Y/X_1$ and $T_0 \cong Y/X_0$. Now $\oplus_{j=1}^{n} S_j \rightarrowtail X_i$, but since $\mathrm{Int}_Y(X_0, X_1) = \{0\}$ we conclude that $n = 0$ and that the X_i are $\mathcal{E}$-simple and we are done. $\square$

6.2. The Artin-Wedderburn exact structures for Nakayama algebras

We characterise all Artin-Wedderburn exact structures for any Nakayama algebra Λ. It turns out they are exactly the Jordan-Hölder exact categories for mod Λ, the category of finitely generated left Λ-modules.

A finite-dimensional algebra Λ is called *Nakayama* if every indecomposable right and left projective Λ-module is uniserial. The representation theory of Nakayama algebras is well-known (see e.g. [1, Chapter V] or [2, Section VI.2]), we recall some details here:

The indecomposable Λ-modules are all uniserial, thus determined by the list of its composition factors from top to socle, which can be represented by a word w in the vertices of the quiver of Λ. Denote the module corresponding to a word w by $[w]$. Equivalently, indecomposable Λ-modules are parametrized by the non-zero paths in the quiver Q of Λ.

If we label the vertices of the path in Q corresponding to the indecomposable module $[w]$ as

$$c \to c+1 \to \cdots \to d-1 \to d$$

then we denote the module $[w]$ also by $[w] = [c, d]$. In this case, the Auslander-Reiten quiver of Λ contains a subquiver of the form described in Fig. 1 where we label the Auslander-Reiten sequences $\eta_{[c,d-1]}$ in

$\mathcal{A} = \operatorname{mod} \Lambda$ by the module $[c, d-1]$ where the sequence ends; the sequence starts in the Auslander-Reiten translate $\tau[c, d-1] = [c+1, d]$. For indecomposables $[w]$ and $[w']$, the space

$$\operatorname{Ext}^1_\Lambda([w], [w'])$$

is at most one-dimensional, and a basis can be given by the following non-split short exact sequences: If $[ww']$ is indecomposable, then a basis is given by

$$\eta_{w,w'} : 0 \longrightarrow [w] \longrightarrow [ww'] \longrightarrow [w'] \longrightarrow 0.$$

If $w = uv$ and $w' = vt$ such that $[uvt]$ is indecomposable, then a basis is given by

$$\eta_{w,w'} : 0 \longrightarrow [w] \longrightarrow [v] \oplus [uvt] \longrightarrow [w'] \longrightarrow 0.$$

We refer to $[ww']$ respectively $[uvt]$ as the *top module* in the extension $\eta_{w,w'}$. Thus for the Auslander-Reiten sequence $\eta_{[c,d-1]}$, the top module is $[c, d]$. The description of the indecomposables and the Auslander-Reiten sequences in $\mathcal{A}$ can be obtained from [12] for the more general case of string algebras, and the basis for the Ext^1-spaces is given in [7] for gentle algebras.

Our first step is to give a more precise description of an exact structure on $\mathcal{A}$ using the Auslander-Reiten sequences it contains.

Theorem 6.9. *Let $\mathcal{B}$ be a set of Auslander-Reiten sequences in $\mathcal{A}$ and $\mathcal{E} = \mathcal{E}(\mathcal{B})$ be the corresponding exact structure on $\mathcal{A}$. Then the short exact sequence $\eta_{w,w'} \in \mathcal{E}$ if and only if the Auslander-Reiten sequence $\eta_{[u]}$ belongs to $\mathcal{B}$ whenever there is a non-zero morphism from $[w]$ to $\tau[u]$ and from $[u]$ to $[w']$.*

Proof. Necessity follows directly from axioms (A2) and $(\text{A2})^{op}$: One can easily verify that forming push-outs and pull-backs of the given exact sequence $\eta_{w,w'}$ along the morphisms from $[w]$ to $\tau[u]$ and from $[u]$ to $[w']$ yields the desired Auslander-Reiten sequences, which thus belong to $\mathcal{E}$.

Sufficiency follows from the fact that exact structures $\mathcal{E}$ on $\mathcal{A}$ correspond to *closed* subfunctors of the bifunctor $\operatorname{Ext}^1(-,-)$ on $\mathcal{A}$, see [14]. Auslander-Reiten theory implies that the socle of $\operatorname{Ext}^1(-,-)$ is given by the Auslander-Reiten sequences, and a closed subfunctor $\mathcal{E} = \mathcal{E}(\mathcal{B})$ is uniquely determined by its socle $\mathcal{B}$, see [3]. We show in [3] that $\mathcal{E} = \mathcal{E}(\mathcal{B})$ is the maximal subfunctor of $\operatorname{Ext}^1(-,-)$ whose socle is $\mathcal{B}$, therefore the sequence $\eta_{w,w'}$ (which induces the socle elements in $\mathcal{B}$ as we showed above when discussing necessity) must belong to $\mathcal{E} = \mathcal{E}(\mathcal{B})$. Here we indicate how to verify this directly from the axioms and leave the details to the reader.

Consider first the case where $[w] = \tau[u]$ and there is an arrow in the Auslander-Reiten quiver from $[u]$ to $[w']$:

$$\begin{tikzcd}
 & & {[a]} \arrow[dr] & \\
 & {[b]} \arrow[ur] \arrow[dr] & & {[w']} \arrow[ll, dotted, "\eta_{[w']}"'] \\
{[w]} \arrow[ur] \arrow[dr] & & {[u]} \arrow[ur] \arrow[ll, dotted, "\eta_{[u]}"] & \\
 & {[c]} \arrow[ur] & &
\end{tikzcd}$$

By assumption, the Auslander-Reiten sequence $\eta_{[u]}$ belongs to $\mathcal{B}$ since there is an irreducible morphism from $[u]$ to $[w']$. Moreover, since the identity is a non-zero morphism, the Auslander-Reiten sequence $\eta_{[w']}$

also belongs to $\mathcal{B}$. We wish to apply axiom (A1) of an exact structure to this situation, however the monos from $\eta_{[u]}$ and $\eta_{[w']}$ cannot be composed directly, only when considering the direct sum of the split exact sequence $(id_{[c]}, 0)$ with $\eta_{[w']}$ this becomes possible. It turns out that the composition of the mono from $\eta_{[u]}$ with the mono of the short exact sequence $\eta_{[w']} \oplus (id_{[c]}, 0)$ yields the mono of the short exact sequence $\eta_{w,w'} \oplus (id_{[c]}, 0)$, which belongs to $\mathcal{E}$ by axiom (A1). Then [10, Corollary 2.18] shows that $\eta_{w,w'} \in \mathcal{E}$. To finish the proof, proceed by induction along paths from $[w]$ to $\tau[w']$ and from $\tau^{-1}[w]$ to $[w']$. $\square$

Remark 6.10. As Λ is Nakayama, the poset of submodules of an indecomposable $[c, d]$ is totally ordered. In particular, for any exact structure $\mathcal{E}$ on $\mathcal{A}$, the poset of proper $\mathcal{E}$-subobjects $\mathcal{S}_{[c,d]}$ is also totally ordered. Hence all indecomposable non $\mathcal{E}$-simple objects have a unique maximal $\mathcal{E}$-subobject. Moreover, all ($\mathcal{E}$-)subobjects of $[c, d]$ are of the form $[x, d]$ for some $c \leq x \leq d$, whereas all quotients are of the form $[c, y]$ for some $c \leq y \leq d$, see Fig. 1.

Now we may classify all Artin-Wedderburn exact structures on $\mathcal{A} = \operatorname{mod} \Lambda$ when Λ is Nakayama.

Theorem 6.11. *Let $\mathcal{B}$ be a set of Auslander-Reiten sequences in $\mathcal{A} = mod\,\Lambda$ and $\mathcal{E} = \mathcal{E}(\mathcal{B})$ be the corresponding exact structure on $\mathcal{A}$. Then $\mathcal{E}$ is Artin-Wedderburn if and only if for all Auslander-Reiten sequences $\eta_{[w]} \in \mathcal{B}$ the top module of this sequence is not $\mathcal{E}$-simple.*

Proof. We use the notation from Fig. 1. To simplify the presentation of the proof, we introduce phantom zero objects $[x, y] = 0$ whenever $x > y$. In this notation, all Auslander-Reiten sequences

$$\eta_{[c,d-1]} : [c + 1, d] \rightarrowtail [c, d] \oplus [c + 1, d - 1] \twoheadrightarrow [c, d - 1]$$

have two middle terms, with top module $[c, d]$, and where $[c + 1, d - 1]$ denotes the zero object when $c + 1 > d - 1$.

We first suppose that there exists an Auslander-Reiten sequence $\eta_{[c,d-1]}$ in $\mathcal{B}$ such that the top module $[c, d]$ is $\mathcal{E}$-simple, and we show that this implies $\mathcal{E}$ being not Artin-Wedderburn. Let $y \leq d - 1$ be such that $[c, y]$ is $\mathcal{E}$-simple and $\eta_{[c,j]} \in \mathcal{B}$ for all $j \in (y, d - 1]$. Such a y always exists. Indeed, if $[c, d - 1]$ is $\mathcal{E}$-simple then we take $y = d - 1$. Else, let $[c, y]$ be an $\mathcal{E}$-simple factor module of $[c, d - 1]$, then y satisfies the required conditions by Theorem 6.9.

Now, let $x \geq c$ be maximal such that there is an indecomposable non-split short exact sequence in $\mathcal{E}$ of the form

$$[x, d] \rightarrowtail [c, d] \oplus [x, y] \twoheadrightarrow [c, y]. \tag{3}$$

Note that $[x, y] \ncong 0$ by the assumption that $[c, d]$ is $\mathcal{E}$-simple. If $[x, y]$ is $\mathcal{E}$-simple then this sequence shows that the implication (AW2) $\Rightarrow$ (AW1) does not hold. Suppose that $[x, y]$ is not $\mathcal{E}$-simple and let $[w, y]$ be its unique maximal $\mathcal{E}$-subobject, note that $w > x$. Thus

$$\operatorname{rad}_{\mathcal{E}}([c, d] \oplus [x, y]) \subseteq \operatorname{Int}_{[c,d]\oplus[x,y]}([c, d] \oplus [w, y], [x, d]).$$

Observe that the $\mathcal{E}$-subobjects of $[x, d]$ are of the form $[i, d]$ with $i \in (x, d]$ and the only possible indecomposable $\mathcal{E}$-subobjects of $[c, d] \oplus [w, y]$ are of the form $[j, y]$ with $j \in (w, y]$ or $[w, d]$. We deduce that, if $\operatorname{rad}_{\mathcal{E}}([c, d] \oplus [x, y]) \neq \{0\}$ then $[w, d]$ is an $\mathcal{E}$-subobject of $[c, d] \oplus [w, y]$. But this is a contradiction to the maximality of x, thus the implication (AW3) $\Rightarrow$ (AW1) does not hold.

For the converse, consider a non-split $\mathcal{E}$-sequence. Since $\operatorname{Ext}^1(-, -)$ is an additive bifunctor, it suffices to consider short exact sequences with indecomposable end terms, which are for Nakayama algebras of the form

$$[c,d] \rightarrowtail [a,d] \oplus [b,c] \twoheadrightarrow [a,b]$$

where $[b,c]$ may denote the zero object. Note that the inequalities $a \leq c-1 \leq b \leq d-1$ must hold. By assumption and Theorem 6.9; $[a,d]$ is not $\mathcal{E}$-simple. Thus, since $\mathcal{A}$ is Krull-Schmidt, $[a,d]\oplus[b,c]$ is not $\mathcal{E}$-semisimple. This shows that the implication (AW2) $\Rightarrow$ (AW1) holds. It remains to show that $\mathrm{rad}_{\mathcal{E}}([a,d]\oplus[b,c]) \neq \{0\}$. Observe that

$$\mathcal{S}_{[a,d]\oplus[b,c]} = \Big\{[c,d],\ [i,d]\oplus[c,b],\ [a,d]\oplus[j,b] \text{ with } [i,d]\in\mathcal{S}_{[a,d]},\ [j,b]\in\mathcal{S}_{[c,b]}\Big\}$$

Let $[x,d] \rightarrowtail [a,d]$ be the unique maximal $\mathcal{E}$-subobject which exists by assumption that the top module $[a,d]$ is not $\mathcal{E}$-simple, and let $[y,b] \rightarrowtail [c,b]$ be the unique maximal $\mathcal{E}$-subobject of $[c,b]$ if it exists or the identity if not. Then

$$\mathrm{Max}(\mathcal{S}_{[a,d]\oplus[c,b]}) \subseteq \Big\{[c,d], [x,d]\oplus[c,b],\ [a,d]\oplus[y,b]\Big\}.$$

First suppose that $x \geq c$. Then, as $a < c$ by Theorem 6.9, $[x,d] \rightarrowtail [c,d]$ and

$$\mathrm{rad}_{\mathcal{E}}\Big([a,d]\oplus[c,b]\Big) \supseteq \mathrm{Int}_{[a,d]\oplus[c,b]}\Big([c,d],[x,d]\oplus[c,b],\ [a,d]\oplus[y,b]\Big) = \Big\{[x,d]\Big\} \neq 0.$$

Now suppose that $x < c$. Then $b > x$ as $c-1 \leq b$. Now

$$\mathrm{rad}_{\mathcal{E}}\Big([a,d]\oplus[c,b]\Big) \supseteq \mathrm{Int}_{[a,d]\oplus[c,b]}\Big([c,d],[x,d]\oplus[c,b],\ [a,d]\oplus[y,b]\Big) = \Big\{[x,d]\oplus[y,b]\Big\} \neq 0$$

and we are done. $\square$

Note that Enomoto studies in [18] the Jordan-Hölder property for torsion-free classes in the module category of a Nakayama algebra endowed with the maximal exact structure. We investigate now when $\mathcal{A} = \mathrm{mod}\,\Lambda$ with any exact structure $\mathcal{E}$ is Jordan-Hölder:

Theorem 6.12. *Let Λ be a Nakayama algebra, and denote $\mathcal{A} = mod\,\Lambda$. Then an exact category $(\mathcal{A},\mathcal{E})$ is $\mathcal{E}$-Artin-Wedderburn precisely when it is Jordan-Hölder.*

Proof. The $\mathcal{E}$-Artin-Wedderburn categories are Jordan-Hölder by Theorem 6.8. Conversely, assume that $(\mathcal{A},\mathcal{E}) = (\mathcal{A},\mathcal{E}(\mathcal{B}))$ is Jordan-Hölder. By [18, Theorem 4.13], we know that the number s of $\mathcal{E}$-simple objects equals the number p of $\mathcal{E}$-projective indecomposable objects. Every non-$\mathcal{E}$-projective indecomposable admits an Auslander-Reiten sequence in $\mathcal{B}$, therefore

$$s = p = |ind(\mathcal{A})| - |\mathcal{B}|$$

where $ind(\mathcal{A})$ denotes the (isoclasses of) indecomposables in $\mathcal{A}$. We conclude

$$|ind(\mathcal{A})| = |\mathcal{B}| + s,$$

and parametrise the set of indecomposables by the $\mathcal{E}$-simples together with the top module for every Auslander-Reiten sequence. Clearly this top module cannot be $\mathcal{E}$-simple in this case, thus by Theorem 6.11, $(\mathcal{A},\mathcal{E})$ is $\mathcal{E}$-Artin-Wedderburn. $\square$

7. The length function

In this section, we consider a Jordan-Hölder exact category $(\mathcal{A}, \mathcal{E})$ and we study the $\mathcal{E}$-Jordan-Hölder length function $l_\mathcal{E}$ that the $\mathcal{E}$-Jordan-Hölder theorem allows us to define over the set $Obj\mathcal{A}$ of isomorphism classes of objects. Throughout, $(\mathcal{A}, \mathcal{E})$ denotes an $\mathcal{E}$-finite essentially small Jordan-Hölder exact category. To simplify notation, we will not distinguish here between the isomorphism class $[X]$ of an object X of $\mathcal{A}$ and the object X.

Definition 7.1. We define the $\mathcal{E}$-Jordan-Hölder length $l_\mathcal{E}(X)$ of an object X in $\mathcal{A}$ as the length of an $\mathcal{E}$-composition series of X. That is $l_\mathcal{E}(X) = n$ if and only if there exists an $\mathcal{E}$-composition series

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X.$$

We say in this case that X is $\mathcal{E}$-finite. If no such bound exists, we say that X is $\mathcal{E}$-infinite. Clearly, isomorphic objects have the same length, and therefore this definition gives rise to a length function $l_\mathcal{E} : Obj\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ defined on isomorphism classes.

Now we prove some corollaries of the $\mathcal{E}$-Jordan-Hölder theorem:

Corollary 7.2. *Let*

$$X \rightarrowtail Z \twoheadrightarrow Y$$

be an admissible short exact sequence of finite length objects. Then

$$l_\mathcal{E}(Z) = l_\mathcal{E}(X) + l_\mathcal{E}(Y).$$

Proof. We know that X is a subobject of Z and that $Y \cong Z/X$. We consider the following composition series of X and Y

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X$$

$$0 = Z_0/X \rightarrowtail Z_1/X \rightarrowtail \dots \rightarrowtail Z_{l-1}/X \rightarrowtail Z_l/X \cong Y$$

where we use the fourth $\mathcal{E}$-isomorphism theorem (Proposition 3.8) to obtain the particular structure for the composition series of Y. Since

$$(Z_{i+1}/X)/(Z_i/X) \cong (Z_{i+1}/Z_i)$$

by [10, Lemma 3.5], the following is a composition series of Z:

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X = Z_0 \overset{i}{\rightarrowtail} \\ \overset{i}{\rightarrowtail} Z_1 \rightarrowtail \dots \rightarrowtail Z_{l-1} \rightarrowtail Z_l = Z$$

Thus

$$l_\mathcal{E}(Z) = n + l = l_\mathcal{E}(X) + l_\mathcal{E}(Y) \quad \square$$

We show now that the function $l_\mathcal{E}$ is a *length function* in the sense of [28]:

Definition 7.3. A *measure for a poset* $\mathcal{S}$ is a morphism of posets $\mu : \mathcal{S} \to P$ where $(\mathcal{P}, \leq)$ is a totally ordered set. A measure μ is called a *length function* when $\mathcal{P} = \mathbb{N}$ with the natural order.

Theorem 7.4. *The function $l_{\mathcal{E}}$ of an $\mathcal{E}$-finite Jordan-Hölder exact category $(\mathcal{A}, \mathcal{E})$ is a* length function *for the poset $Obj\mathcal{A}$.*

Proof. The function $l_{\mathcal{E}} : Obj\mathcal{A} \to \mathbb{N}$ is defined on the set $Obj\mathcal{A}$, which is partially ordered by the $\mathcal{E}$-subset relation $X \subset_{\mathcal{E}} Y$, see [8, Proposition 6.11]. Moreover, consider X and Y in $Obj\mathcal{A}$ with $X \subset_{\mathcal{E}} Y$. Then by Corollary 7.2 we have

$$l_{\mathcal{E}}(X) \leq l_{\mathcal{E}}(Y),$$

so $l_{\mathcal{E}}$ is a morphism of posets. $\square$

As a consequence of the previous result, an $\mathcal{E}$-finite object is an object with $\mathcal{E}$-finite length.

Proposition 7.5. *(**$\mathcal{E}$-Hopkins-Levitzki theorem**) An object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian if and only if it has an $\mathcal{E}$-finite length.*

Proof. For an $\mathcal{E}$-finite object X of length $l_{\mathcal{E}}(X) = n \in \mathbb{N}$, the composition series is of length n. Thus any increasing or decreasing sequence of $\mathcal{E}$-subobjects of X must become stationary and X is $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian.

Conversely, let X be an $\mathcal{E}$-Artinian and $\mathcal{E}$-Noetherian object. Then any composition series ending with X has to be of finite length. So X is $\mathcal{E}$-finite. $\square$

Remark 7.6. Note that a length function for exact categories in general was studied in [8, Section 6]. The notion there was defined as maximum over all lengths of an $\mathcal{E}$-composition series; in the case of an $\mathcal{E}$-Jordan-Hölder category all composition series of an object have the same length, so the definition we use here is compatible with the one from [8].

Definition 7.7. We denote by $(Ex(\mathcal{A}), \subseteq)$ the poset of exact structures $\mathcal{E}$ on $\mathcal{A}$, where the partial order is given by containment $\mathcal{E}' \subseteq \mathcal{E}$. This *containment* partial order is studied in [8, Section 4].

We conclude by noting that, similarly to [8, Lemma 8.1], the $\mathcal{E}$-Jordan Hölder length function can only decrease under reduction of exact structures:

Proposition 7.8. *If $\mathcal{E}$ and $\mathcal{E}'$ are exact structures on $\mathcal{A}$ such that $\mathcal{E}' \subseteq \mathcal{E}$, then $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ for all objects X in $\mathcal{A}$.*

Proof. Let us consider an $\mathcal{E}'$-composition series of ending by X

$$0 = X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{n-1}} X_{n-1} \xrightarrow{i_n} X = X_n$$

where $l_{\mathcal{E}'}(X) = n$. Since $\mathcal{E}' \subseteq \mathcal{E}$, all these pairs (i_j, d_j) will also be in $\mathcal{E}$. So the $\mathcal{E}'$-composition series is also an $\mathcal{E}$-composition series and therefore by definition $l_{\mathcal{E}}(X) \geq n$. $\square$

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ANNEXE C

ON THE LATTICES OF EXACT AND WEAKLY EXACT STRUCTURES

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Abstract. We initiate in this article the study of *weakly exact structures*, a generalisation of Quillen exact structures. We introduce weak counterparts of one-sided exact structures and show that a left and a right weakly exact structure generate a weakly exact structure. We further define weakly extriangulated structures on an additive category and characterize weakly exact structures among them.

We investigate when these structures on $\mathcal{A}$ form lattices. We prove that the lattice of substructures of a weakly extriangulated structure is isomorphic to the lattice of topologizing subcategories of a certain functor category. In the idempotent complete case, we characterise the lattice of all weakly exact structures and we prove the existence of a unique maximal weakly exact structure.

We study in detail the situation when $\mathcal{A}$ is additively finite, giving a module-theoretic characterization of closed sub-bifunctors of Ext^1 among all additive sub-bifunctors.

Contents

1. Introduction and historical remarks

In this work we are studying *all* the additive sub-bifunctors of the first extension functor Ext on an additive category. They are associated to what we introduce as *the weakly exact structures*, which generalize Quillen's exact structures. The extension functor Ext was introduced by Reinhold Baer in 1934 for the special case of abelian groups and then generalized and studied by Cartan and Eilenberg [CE56], using projective or injective resolutions. In [Buch55], Buchsbaum shows the existence of the functor Ext for an exact category having enough projectives or enough injectives. In his work the use of projectives and injectives is essential.

The need for an abstract homology theory, independent of projective objects and projective resolutions, became apparent when it was realized that the category of sheaves and sheaf homomorphisms over a topological space is an exact category, however sufficiently many projectives do not exist in that category. In an unpublished work, A. Grothendieck has established the existence of sufficiently many injectives and defined the functor Ext on the category of sheaves. The ideas of *relative homological algebra for categories of modules* have subsequently been formulated by G. Hochschild. In [H58], he discusses the analogue of the Tor and Ext functors of Cartan and Eilenberg, but applicable to a module theory that is relativized with respect to a given subring of the basic ring of operators. The special class of extensions he considered is the class of extensions which split over a given subring of the ring of operators.

In [Y54], N. Yoneda studied the groups $Ext^n_\Lambda(A, B)$ and $Tor^n_\Lambda(A, B)$ defined by Cartan and Eilenberg, and explicitly described their characteristic properties. He proved *the classification theorem*, a one-to-one correspondence between the equivalence classes of the $n-$fold extensions of B by A (exact sequences of length n, from B to A), and the elements of the abelian group $Ext^n_\Lambda(A, B)$. Following Yoneda's ideas, D. Buchsbaum defines in [Buch59] the extension functor Ext *without* using the projective and the injective objects of the category. The idea of *relative homological algebra for abstract categories* is introduced in the works of Buchsbaum [Buch59] and Heller [He58], by selecting of a class of extensions or, equivalently, a class of monomorphisms and epimorphisms. This class of extensions is used either to construct resolutions of objects of the category, and so obtain the values of derived functors as homology objects, or to construct the relative derived functors of Hom as equivalence classes of extensions.

Later in [BuHo61], M.C.R. Butler and G. Horrocks study relative homological algebra, in the context of abelian categories. They study how the derived functors behave under reduction of the exact structure, that is, they discuss the relation between the derived functors constructed from two classes of extensions one of which contains the other. M.Auslander and Ø.Solberg discuss in part one [AS93] of their more recent series of works on the topic how to apply relative homological algebra to representation theory. Then later in part two [AS05] they develop a general theory of relative cotilting modules for artin algebras.

In these first papers on relative homological algebra, a mix of structures has been considered that correspond to what is nowadays called an exact structure: on one hand classes of morphisms satisfying certain properties ("h.f.class"), on the other hand certain ("closed") subfunctors of Ext. The authors considered also a weaker notion, an *f.class*, which omits the condition on admissible monics and epics to be closed under composition. This weaker notion corresponds to an additive subfunctor of Ext. It has been studied more recently in the work of Fu, Guil Asensio, Herzog and Torrecillas [FGHT13], which extended the theory of approximation in the relative homological algebra to the setup of morphisms (more precisely, ideals in the category) rather than objects. They demonstrated the need to study the more general notion of *f.class* by considering examples such as the Auslander–Reiten phantom morphisms. [FGHT13] work in the context of a given exact category $(\mathcal{A}, \mathcal{E})$, and consider links between ideals of morphisms and additive sub-bifunctors of the extension functor $Ext_{\mathcal{E}}$ associated to $\mathcal{E}$. This work has been further extended by Breaz and Modoi [BM15] to the context of extension-closed subcategories $\mathcal{A}$ of a triangulated category $\mathcal{T}$ and restrictions of sub-bifunctors of $\mathcal{T}(-, -[1])$ on $\mathcal{A}$.

The "stand alone" concept of an exact structure as a class of short exact sequences in an additive category $\mathcal{A}$ satisfying certain axioms has been laid out by Quillen in [Qu73], however requiring $\mathcal{A}$ to be embedded into an abelian category. The independent version of these axioms was formulated by Keller in [Ke90], see also [GR92]. It allows to develop methods from homological algebra, and define derived categories, see [Ne90, Ke91]. Note that there exist different independent notions of "exact categories", like the "Barr-exact categories" or "effective regular categories", not to be confused with the one we consider in our work. The comparison to sub-bifunctors of Ext^1 has been re-considered in [AS93] and then in [DRSS], with applications to exact structures originating from one-point extensions, a special case of exact structures associated with bimodule problems in [BrHi]. However, the lack of a unique maximum extension-functor for arbitrary additive categories was a limiting factor in these studies. If $\mathcal{A}$ has kernels and cokernels, the existence of a unique maximal exact structure was first proved by Sieg and Wegner [SW11]. Crivei [Cr11] extended the result to additive categories for which every split epimorphism has a kernel, and finally Rump [Ru11] showed that any additive category admits a unique maximal exact structure $\mathcal{E}_{\max}$. In [BHLR] a study of the family of all exact structures $\mathbf{Ex}(\mathcal{A})$ on an additive category $\mathcal{A}$ was initiated. The existence of a unique maximum exact structure allows to turn $\mathbf{Ex}(\mathcal{A})$ into a complete bounded lattice. On the side of bifunctors, this amounts to studying all closed sub-bifunctors of a unique maximum bifunctor $\mathbb{E}_{\max}$ which corresponds to the exact structure $\mathcal{E}_{\max}$. It is natural, on the bifunctor side, to extend the study to *all* additive sub-bifunctors, which in turn raises the question to which structure of exact sequences they correspond. Moreover it is very interesting to generalise and prove the existence of a maximal weakly exact structure and to compare it with the maximal exact structure.

In this work we introduce the notion of a *weakly exact structure* on an additive category $\mathcal{A}$. It provides a conceptualization of the notion of f.classes studied in [Buch55] and the notion of additive subfunctors of an extension functor $\mathcal{E}$ studied in [AS93, DRSS, FGHT13]. We establish the existence of a unique maximal weakly exact structure provided the additive category $\mathcal{A}$ is weakly idempotent complete. This in turn allows to show that all the weakly exact structures on $\mathcal{A}$ form a lattice. When the underlying category $\mathcal{A}$ is additively finite, this lattice is a finite length modular lattice, a class of lattices studied recently by Haiden, Katzarkov, Kontsevich and Pandit in [HKKP] in connection with weight filtrations and the notion of semi-stability.

We introduce in Section 3 the class $\mathbf{Wex}(\mathcal{A})$ of all weakly exact structures on an additive category $\mathcal{A}$. It turns out that, despite the fact that weakly exact structures are not closed under compositions, some of the properties of exact structures are still valid, in particular, every weakly exact structure satisfies Quillen's obscure axiom, see Proposition 3.8. Similar to exact structures, it is sometimes beneficial to dissect the set of axioms into two parts, leading to the notion of left and right weakly exact structures. We show that any pair of a left and a right weakly exact structures gives rise to a weakly exact structure, and that all such structures arise in that way.

It is known and proved in [Cr11], that the stable exact structure $\mathcal{E}_{sta}$ forms the maximal exact structure on any weakly idempotent complete category. A generalization of this result is given in [Cr12], where they characterise the additive category admitting $\mathcal{E}_{sta}$ as maximal exact structure.

In this work we generalise these results by proving that a unique maximal weakly exact structure exists on any weakly idempotent complete category, and is given by the stable short exact sequences. We then also deduce a characterisation of the additive categories where the stable short exact sequences forms the unique weakly exact structure and coincides with the maximal exact structure.
We also consider the interval $\mathbf{Wex}(\mathcal{E}_{max}) := [\mathcal{E}_{min}, \mathcal{E}_{max}] \subseteq \mathbf{Wex}(\mathcal{A})$ and we study the weakly exact structures that are included in the unique maximal exact structure $\mathcal{E}_{max}$. Given a weakly exact structure $\mathcal{W}$ on $\mathcal{A}$, constructing the group $\mathbb{W}$ of $\mathcal{W}-$extensions yields a map Φ to category of bifunctors from $\mathcal{A}$ to abelian groups:

$$\Phi : \mathbf{Wex}(\mathcal{A}) \longrightarrow \mathbf{BiFun}(\mathcal{A})$$
$$\mathcal{W} \longmapsto \mathbb{W} = \mathrm{Ext}^1_{\mathcal{W}}(-,-).$$

This function Φ induces lattice isomorphisms

$$\begin{array}{ccc} \mathbf{Wex}(\mathcal{E}_{max}) & \longleftrightarrow & \mathbf{BiFun}(\mathbb{E}_{max}) \\ \cup & & \cup \\ \mathbf{Ex}(\mathcal{A}) & \longleftrightarrow & \mathbf{CBiFun}(\mathcal{A}) \end{array}$$

where $\mathbf{CBiFun}(\mathcal{A})$ denotes the subclass of closed sub-bifunctors of $\mathbb{E}_{max}$.
Note that $\mathbf{Ex}(\mathcal{A})$ is *not* a *sublattice* of $\mathbf{Wex}(\mathcal{E}_{max})$, even if it is a subposet: the join operations we consider on these sets are different, as we illustrate by an example in Section 4.3.

When the underlying category $\mathcal{A}$ is additively finite and Krull-Schmidt, it is known that the lattice $\mathbf{Ex}(\mathcal{A})$ is boolean, with each object $\mathbb{E}(S)$ determined by the choice of a set S of Auslander-Reiten sequences. The larger lattice $\mathbf{Wex}(\mathcal{A})$ however is not boolean, and it is interesting to characterise the members of $\mathbf{Ex}(\mathcal{A})$ in module-theoretic terms, that is, describe the closed sub-bimodules of $\mathbb{E}_{max}$. We show that, when viewed as bimodules over the Auslander algebra of $\mathcal{A}$, elements in $\mathbf{Ex}(\mathcal{A})$ can be characterized as follows: For every set S of Auslander-Reiten sequences, the closed bimodule $\mathbb{E}(S)$ of $\mathbb{E}_{max}$ introduced above is the maximal submodule of $\mathbb{E}_{max}$ whose socle is S.

In order to find a general and simultanious way to give proofs of various statements concerning exact and triangulated categories at the same time, Nakaoka and Palu [NP19] studied additive bifunctors $\mathcal{E} : \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Ab}$ equipped with certain extra data called a *realization*. They found a set of axioms on triples consisting of an additive category, a bifunctor and a realization that unifies the axioms of exact and of triangulated categories. They called such structures *extriangulated*. Extensions in exact categories are realized by "admissible" kernel-cokernel pairs. In an extriangulated category this role is played by pairs of composable morphisms f, g where f is a *weak kernel* of g and g is a *weak cokernel* of f. Moreover, Nakaoka and Palu characterized all triples that define exact structures, in other words, closed additive sub-bifunctors of $\mathbb{E}_{max}$. Hershend, Liu and Nakaoka [HLN] introduced $n-$*exangulated* structures and proved that the choice of a $1-$exangulated structure on an additive category is equivalent to the choice of an extriangulated structure. The set of axioms of $1-$exangulated structures is slightly different from that of extriangulated categories. In Section 5, we consider $1-$exangulated categories with one of the axioms removed. We prove that such *weakly* $1-$*exangulated*, or *weakly extriangulated* structures naturally generalize weakly exact structures we defined earlier. We also show that *almost exact structures* on extension closed subcategories of triangulated categories, which were considered by Breaz and Modoi in [BM15], are weakly extriangulated.

For a finite-dimensional algebra Λ, Buan [Bu01] studied closed sub-bifunctors of the bifunctor Ext^1_Λ on the category $\mathbf{mod}\,\Lambda$. He proved that they correspond to certain Serre subcategories of the category of finitely presented additive functors $(\mathbf{mod}\,\Lambda)^{op} \to \mathrm{Ab}$ (i.e. of *finitely presented* modules over $\mathbf{mod}\,\Lambda$), defined as categories of *contravariant defects* in works of Auslander [A66, A78]. This result was later extended to exact structures on additive categories in [En18], see also [FG20]. We note that the definition of contravariant defects naturally extends to the setting of weakly exact structures. Ogawa [Og19] defined contravariant defects in the setting of extriangulated categories, and we further extend this notion to the framework of weakly extriangulated categories. By adapting arguments of Ogawa and Enomoto [En20], we prove that the category of defects of a weakly extriangulated structure on an additive category $\mathcal{A}$ is *topologizing* (in the sense of Rosenberg [Ros]) in the category $\mathbf{coh}(\mathcal{A})$ of coherent right $\mathcal{A}-$modules. That means that it is closed under subquotients and finite coproducts.

Given a weakly extriangulated structure, all its substructures are uniquely characterized by their categories of defects, and each topologizing subcategory of a given category of defects defines a weakly extriangulated substructure. Weakly extriangulated substructures of a weakly exact structure are necessarily weakly exact. Thus, whenever we know that an additive category $\mathcal{A}$ admits a unique maximal weakly exact structure, we can classify all weakly exact structures on $\mathcal{A}$ in terms of topologizing structures in a certain abelian category. As explained above, this covers all weakly idempotent complete additive categories.

Topologizing subcategories of an abelian category form a lattice. Topologizing subcategories of the (not necessarily abelian) category of defects of a weakly extriangulated structure on $\mathcal{A}$ also form a lattice, which is an interval in the lattice of all topologizing subcategories of $\mathbf{coh}(\mathcal{A})$. Note that Serre subcategories form a subposet, but not a sublattice of this lattice. Weakly extriangulated substructrures of a weakly extriangulated structure also form a natural lattice, extending the lattice of weakly exact structues. We establish, in the last section of this work, lattice isomorphisms between these several lattices.

We summarize, in the following figure, the lattice isomorphisms between the following lattice structures, for an additive category satisfying the conditions of 3.17:

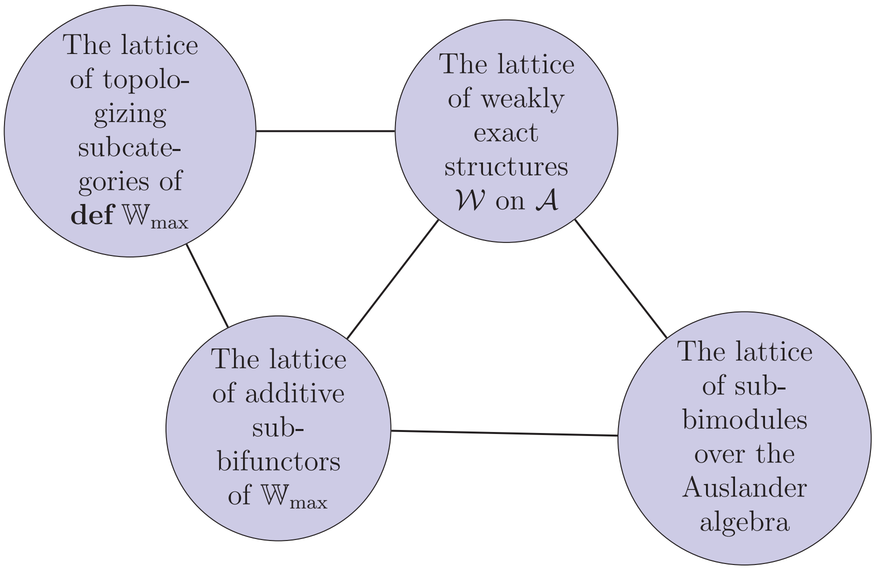

Figure 1: Isomorphisms of lattices

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3. Weakly exact and exact structures

We introduce in this section the central topic of this paper, weakly exact structures, and study some of their properties.

3.1. **Exact structures.** We recall the definition of an exact structure on an additive category given by Quillen in [Qu73], using the terminology of [Ke90], see also [GR92]. We refer to [Bü10] for an exhaustive introduction to exact categories.

We fix an additive category $\mathcal{A}$ throughout this section. The notion of short exact sequence is specified to be a kernel-cokernel pair (i, d), that is, a pair of composable morphims such that i is kernel of d and d is cokernel of i. An exact structure on $\mathcal{A}$ is then given by a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathcal{A}$ satisfying certain axioms which we recall below. We call *admissible monic* a morphism i for which there exists a morphism d such that $(i, d) \in \mathcal{E}$. An *admissible epic* is defined dually. Note that admissible monics and admissible epics are referred to as inflation and deflation in [GR92], respectively. We depict an admissible monic by $\rightarrowtail$ and an admissible epic by $\twoheadrightarrow$. The pair $(i, d) \in \mathcal{E}$ is referred to as *admissible short exact sequence*, or *short exact sequence in $\mathcal{E}$*.

Definition 3.1. An *exact structure* $\mathcal{E}$ on $\mathcal{A}$ is a class of kernel-cokernel pairs (i, d) in $\mathcal{A}$ which is closed under isomorphisms and satisfies the following axioms:

(E0) For all objects A in $\mathcal{A}$ the identity 1_A is an admissible monic;

$(E0)^{op}$ For all objects A in $\mathcal{A}$ the identity 1_A is an admissible epic;

(E1) The class of admissible monics is closed under composition

$(E1)^{op}$ The class of admissible epics is closed under composition;

(E2) The push-out of an admissible monic $i : A \rightarrowtail B$ along an arbitrary morphism $t : A \to C$ exists and yields an admissible monic s_C:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "t"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, "s_B"] \\ C \arrow[r, tail, "s_C"'] & S. \end{tikzcd}$$

$(E2)^{op}$ The pull-back of an admissible epic h along an arbitrary morphism t exists and yields an admissible epic p_B

$$\begin{tikzcd} P \arrow[r, two heads, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\text{PB}"] & B \arrow[d, "t"] \\ A \arrow[r, two heads, "h"'] & C. \end{tikzcd}$$

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ consisting of an additive category $\mathcal{A}$ and an exact structure $\mathcal{E}$ on $\mathcal{A}$. Note that $\mathcal{E}$ is an exact structure on $\mathcal{A}$ if and only if $\mathcal{E}^{op}$ is an exact structure on $\mathcal{A}^{op}$.

We denote by $(\mathbf{Ex}(\mathcal{A}), \subseteq)$ the poset of exact structures $\mathcal{E}$ on $\mathcal{A}$, where the partial order is given by containment $\mathcal{E}' \subseteq \mathcal{E}$. Note that $\mathbf{Ex}(\mathcal{A})$ need not actually form a set, but by abuse of language, we still use the term poset when $\mathbf{Ex}(\mathcal{A})$ is a class. The poset $(\mathbf{Ex}(\mathcal{A}), \subseteq)$ always contains a unique minimal element, the *split* *exact structure* $\mathcal{E}_{min}$ which is formed by all split exact sequences, that is, sequences isomorphic to

$$A \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} A \oplus B \xrightarrow{[0\ 1]} B$$

(see [Bü10, Lemma 2.7]).

Moreover, every additive category admits a unique maximal exact structure $\mathcal{E}_{max}$, see [Ru11, Corollary 2]. When the category $\mathcal{A}$ is abelian, then $\mathcal{E}_{max}$ is formed by all short exact sequences in $\mathcal{A}$. The construction is more subtle for other classes of additive categories, we refer to [BHLR, Section 2.4] for a more detailed discussion.

3.2. **Example.** Consider the category $\mathcal{A} = \operatorname{rep} Q$ of representations of the quiver

$$Q: \qquad 1 \longrightarrow 2 \longleftarrow 3$$

Then the Hasse diagram of the poset of exact structures $\mathbf{Ex}(\mathcal{A})$ has the shape of a *cube* (see [BHLR, Example 4.2] for detailed description of the different exact structures on $\mathcal{A}$):

$$\begin{tikzcd}
 & \mathcal{E}_3 \arrow[rr] \arrow[dl] & & \mathcal{E}_{1,3,5} \arrow[dl] \\
\mathcal{E}_{2,3,4} \arrow[rr] & & \mathcal{E}_{max} & \\
 & \mathcal{E}_{min} \arrow[rr] \arrow[dl] \arrow[uu] & & \mathcal{E}_1 \arrow[dl] \arrow[uu] \\
\mathcal{E}_2 \arrow[rr] \arrow[uu] & & \mathcal{E}_{1,2} \arrow[uu] &
\end{tikzcd}$$

Let us mention that by taking other forms of the quiver of type A_3 such as

$$Q: \qquad 1 \longleftarrow 2 \longrightarrow 3$$

or

$$Q: \qquad 1 \longrightarrow 2 \longrightarrow 3$$

we get an isomorphic poset. In fact, $\mathbf{Ex}(\mathcal{A})$ is a Boolean lattice in these cases, with n Auslander-Reiten sequences in $\mathcal{A}$ giving rise to exactly 2^n exact structures and poset structure isomorphic to the power set of the set of Auslander-Reiten sequences in $\mathcal{A}$, see [En18].

3.3. **Weakly exact structures.**

Definition 3.2. Let $\mathcal{A}$ be an additive category. We define a *weakly exact structure* $\mathcal{W}$ on $\mathcal{A}$ as a class of kernel-cokernel pairs (i, d) in $\mathcal{A}$ which is closed under isomorphisms and direct sums, and satisfies the axioms $(E0)$, $(E0)^{op}$,$(E2)$ and $(E2)^{op}$ of Definition 3.1.

This definition provides of the conceptualization of subfunctors of Ext as studied in [FGHT13] in the context of exact categories. We denote by $(\mathbf{Wex}(\mathcal{A}), \subseteq)$ the poset of all weakly exact structures on $\mathcal{A}$, ordered by containment.

Lemma 3.3. $\mathbf{Ex}(\mathcal{A})$ is a subclass of $\mathbf{Wex}(\mathcal{A})$.

Proof. Only the direct sum condition needs to be verified. But this is always satisfied for exact structures, by [Bü10, Proposition 2.9]. □

Remark 3.4. The proof of [Bü10, Proposition 2.9] makes heavy use of axioms $(E1)$ and $(E1)^{op}$, this makes us think that the property of being closed under direct sums does not follow from the remaining axioms for weakly exact structures.

We now state some of the properties for exact structures that also hold for weakly exact structures:

Lemma 3.5. Let $\mathcal{W}$ be a weakly exact structure and let i and i' be admissible monics of $\mathcal{W}$ forming the rows of a commutative square:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "f'"] \\ A' \arrow[r, tail, "i'"] & B' \end{tikzcd}$$

Then the following statements are equivalent:
(i) The square is a push-out.
(ii) $A \overset{\begin{bmatrix} i \\ -f \end{bmatrix}}{\rightarrowtail} B \oplus A' \overset{[f'\ i']}{\twoheadrightarrow} B'$ is a short exact sequence belonging to $\mathcal{W}$.
(iii) The square is both a push-out and a pull-back.
(iv) There exists a commutative diagram with rows being conflations in $\mathcal{W}$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, "1_C"] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

Proof. One can easily verify that the proof of the statement for exact categories in [Bü10, Proposition 2.12] does not use axioms $(E1)$ or $(E1)^{op}$ when it is done in the order $(i) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. □

Remark 3.6. The dual of Lemma 3.5 is also true. For example, the dual of (i) implies (iv) would be: If d and d' are admissible epics of $\mathcal{W}$ and (g, d) is the push-out of (d', g') then the following diagram exists, is commutative and has rows in $\mathcal{W}$:

$$\begin{tikzcd} A \arrow[r, tail, "j"] \arrow[d, "1_A"] & B \arrow[r, two heads, "d"] \arrow[d, "g"] & C \arrow[d, "g'"] \\ A \arrow[r, tail, "j'"] & B' \arrow[r, two heads, "d'"] & C' \end{tikzcd}$$

Commutative squares that are both a pushout and a pullback are called *bicartesian* squares.

Lemma 3.7. Let $\mathcal{W}$ be a weakly exact structure on $\mathcal{A}$. For any morphism of admissible short exact sequences

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & B \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C' \end{tikzcd}$$

in $\mathcal{W}$, there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, equal] \\ A' \arrow[r, tail] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d] & C \arrow[d] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C', \end{tikzcd}$$

where the middle row is also an admissible short exact sequence in $\mathcal{W}$ and the top left and bottom right squares are bicartesian.

Proof. The same proof as in [Bü10, Lemma 3.1] applies here. □

In [FGHT13, Lemma 5], a weaker version of Quillen's obscure axiom is established in the context of weakly exact structures. In fact, the full version is valid in this context, as we now show:

Proposition 3.8. (Quillen's obscure axiom for weakly exact structures) Let $\mathcal{W}$ be a weakly exact structure on an additive category $\mathcal{A}$.

(1) Consider morphisms $A \xrightarrow{i} B \xrightarrow{j} C$ in $\mathcal{A}$, where i has a cokernel and ji is an admissible monic of $\mathcal{W}$. Then i is also an admissible monic of $\mathcal{W}$.

(2) Consider morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, where g has a kernel and gf is an admissible epic of $\mathcal{W}$. Then g is also an admissible epic of $\mathcal{W}$.

Proof. (1) The proof given in [Bü10, Proposition 2.16] also holds for weakly exact categories: Lemma 3.5 is the equivalent of [Bü10, Proposition 2.12]. One step in the proof of [Bü10, Proposition 2.16] is using axiom $(E1)$, but in fact, the composition of an admissible monic with an isomorphism gives an admissible monic because the class $\mathcal{W}$ is closed under isomorphisms.
(2) The proof is done dually. □

Lemma 3.9. The split exact structure $\mathcal{E}_{min}$ forms the unique minimal element of the poset $(\mathbf{Wex}(\mathcal{A}), \subseteq)$.

Proof. The proof of [Bü10, Lemma 2.7] does not use axioms $(E1)$ and $(E1)^{op}$, so the statement of [BHLR, Prop 2.12] applies to weakly exact structures as well. □

3.4. **The left and right weakly exact structures.** In this subsection, we define *left weakly exact structures* and *right weakly exact structures*. We show that their combination gives a weakly exact structure and also that every weakly exact structure can be obtained in this way.
These definitions generalise the left and right exact structures introduced in [BC12, Definition 3.1] and studied in [HR20], and used by Rump in [Ru11].

Definition 3.10. A right weakly exact structure on $\mathcal{A}$ is a class of kernels I which is closed under isomorphisms and satisfies the following properties:

(Id) For all objects X in $\mathcal{A}$ the identity 1_X and the zero monomorphism $0 \longrightarrow X$ are in I.

(P) The push-out of $f : X \longrightarrow Y \in I$ along an arbitrary morphism $h : X \longrightarrow X'$ exists and yields a morphism $f' \in I$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\mathrm{PO}"] & Y \arrow[d, "h'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

(Q) Given $A \xrightarrow{a} B \xrightarrow{b} C$ with $ba \in I$ and a has a cokernel, then a is in I.

(S) I is closed under direct sums of morphisms.

Definition 3.11. A left weakly exact structure on $\mathcal{A}$ is a class of cokernels $\mathcal{D}$ which is closed under isomorphisms and satisfies the following properties:

(Id^{op}) For all objects X in $\mathcal{A}$ the identity 1_X and the zero epimorphism $X \longrightarrow 0$ are in $\mathcal{D}$.

(P^{op}) The pullback of $f : C \longrightarrow F \in \mathcal{D}$ along an arbitrary morphism $h : E \longrightarrow F$ exists and yields a morphism $e \in \mathcal{D}$:

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "e"'] \arrow[dr, phantom, "\mathrm{PB}"] & C \arrow[d, "f"] \\ E \arrow[r, "h"'] & F \end{tikzcd}$$

(Q^{op}) Given $A \xrightarrow{a} B \xrightarrow{b} C$ with $ba \in \mathcal{D}$ and b has a kernel, then b is in $\mathcal{D}$.

(S^{op}) $\mathcal{D}$ is closed under direct sums of morphisms.

Remark 3.12. Note that, contrary to exact structures (see [Bü10, Proposition 2.9]) the properties (S) and $(\mathrm{S})^{op}$ above are not implied by the rest of the properties and we need to add them. These properties are necessary to ensure that we get a structure which is equivalent to an additive sub-bifunctor of Ext^1 as we show in Section 4. The reason behind this is that the Baer sum uses the direct sum of two short exact sequences in its construction.

Theorem 3.13. Let $\mathcal{A}$ be an additive category. A left weakly exact structure $\mathcal{D}$ on $\mathcal{A}$ can be combined with a right weakly exact structure $\mathcal{I}$ to form a weakly exact structure $\mathcal{W}$ given by the short exact sequences $A \xrightarrow{i} B \xrightarrow{d} C$ with $i \in I$ and $d \in \mathcal{D}$.

Proof. We adapt the proof of [Ru11, Theorem 1] to the case of weakly exact structures. $\square$

Proposition 3.14. Every weakly exact structure $\mathcal{W}$ on $\mathcal{A}$ can be constructed from a right weakly exact structure and a left weakly exact structure as in Theorem 3.13. More precisely, if I is the class of admissible monics of a weakly exact structure $\mathcal{W}$ and $\mathcal{D}$ is the class of admissible epics of $\mathcal{W}$, then I is a right weakly exact structure and $\mathcal{D}$ is a left weakly exact structure.

Proof. Let $\mathcal{W}$ be a weakly exact structure on an additive category $\mathcal{A}$. Let I be the class of admissible monics of $\mathcal{W}$ and $\mathcal{D}$ the class of admissible epics of $\mathcal{W}$. First, it is not difficult to show that I and $\mathcal{D}$ are closed under isomorphisms. Second, since $\mathcal{W}$ satisfies $(E0)$ and $(E0)^{op}$, it is clear that I satisfies (Id) and $\mathcal{D}$ satisfies $(\mathrm{Id})^{op}$. Third, by Proposition 3.8, I satisfies (Q) and $\mathcal{D}$ satisfies $(\mathrm{Q})^{op}$. And finally, since $\mathcal{W}$ is closed under direct sums it is clear that I satisfies (S) and $\mathcal{D}$ satisfies $(\mathrm{S})^{op}$. $\square$

3.5. **The maximal weakly exact structure.** We prove the existence of a unique maximal weakly exact structure on any weakly idempotent complete additive catyegory. We also generalise Crivei's characterisation of stable short exact sequences forming the maximal exact structure. We show that, under these conditions, the maximal weakly exact structure coincides with the maximal exact structure.

Definition 3.15. [RW77] A kernel (A, f) is in an additive category $\mathcal{A}$ is called *semi-stable* if for every push-out square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[d, "s_B"] \\ C \arrow[r, "s_C"'] & S \end{tikzcd}$$

the morphism s_C is also a kernel. We define dually a *semi-stable* cokernel. A short exact sequence $A \rightarrowtail^{i} B \twoheadrightarrow^{d} C$ in $\mathcal{A}$ is said to be *stable* if i is a semi-stable kernel and d is a semi-stable cokernel. We denote by $\mathcal{E}_{sta}$ the class of all *stable* short exact sequences.

We generalise Crivei's characterising of the maximum exact structures using the idempotent (or Karoubian) completion $H : \mathcal{A} \to \hat{\mathcal{A}}$ of the category $\mathcal{A}$:

Theorem 3.16. [Cr12, Theorem 3.4] Let $\mathcal{A}$ be an additive category, and let $H : \mathcal{A} \to \hat{\mathcal{A}}$ be its idempotent completion. Then the class $\mathcal{E}_{sta}$ of stable short exact sequences of $\mathcal{A}$ defines an exact structure on $\mathcal{A}$ if and only if $\mathcal{A}$ is closed under pushouts and pullbacks for $(\hat{\mathcal{A}}, \hat{\mathcal{E}}_{max})$. In this case, $\mathcal{E}_{sta}$ is the maximal exact structure on $\mathcal{A}$.

Again, since the class of stable short exact sequences clearly forms the maximal class satisfying (E2) and (E2)^{op}, we get:

Theorem 3.17. Assume that $\mathcal{A}$ is closed under pushouts and pullbacks for $(\hat{\mathcal{A}}, \hat{\mathcal{E}}_{max})$. Then the class of stable short exact sequences forms the unique maximal weakly exact structure on $\mathcal{A}$:

$$\mathcal{W}_{max} = \mathcal{E}_{max} = \mathcal{E}_{sta}$$

Proof. By applying the arguments of the proof [Cr12, Theorem 3.4]. □

Corollary 3.18. Let $\mathcal{A}$ be an weakly idempotent complete additive category, then $\mathcal{A}$ admits a unique maximal weakly exact structure and

$$\mathcal{W}_{max} = \mathcal{E}_{max} = \mathcal{E}_{sta}.$$

Proof. Since a weakly idempotent additive category satisfies the condition of 3.17. □

We refer to [Cr12, Corollary 3.5] for an example of an additive category $\mathcal{A}$ which is not weakly idempotent complete, but satisfies that $\mathcal{A}$ is closed under pushouts and pullbacks for $(\hat{\mathcal{A}}, \hat{\mathcal{E}}_{max})$.

4. Sub-bifunctors and closed sub-bifunctors of Ext^1

We explore in this section the correspondence between weakly exact structures and subfunctors of $\mathrm{Ext}^1_{\mathcal{A}}$.

4.1. **From weakly exact structures to bifunctors.** Let $\mathcal{W}$ be a weakly exact structure on $\mathcal{A}$. The aim of this section is to associate with $\mathcal{W}$ an additive functor to the category of abelian groups

$$\mathbb{W} = \mathrm{Ext}^1_{\mathcal{W}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Ab}.$$

In the following definition we generalise the classical construction for abelian categories, stated in [M65], chapter VII, and formulate it in our context:

Definition 4.1. Define for objects $A, C \in \mathcal{A}$ the set

$$\mathbb{W}(C,A) = \mathrm{Ext}^1_{\mathcal{W}}(C,A) = \left\{ \overline{(i,d)} \quad | \quad A \xrightarrow{i} B \xrightarrow{d} C \quad \in \mathcal{W} \right\},$$

where we denote by $\overline{(i,d)}$ the usual equivalence class of the short exact sequence (i,d). To define the action of the functor $\mathbb{W}$ on morphisms, let $E = (i,d) \in \mathcal{W}$ be a short exact sequence from A to C, and $a : A \longrightarrow A'$ a morphism. Then we define the short exact sequence $aE \in \mathcal{W}$ (using the property (E2) (or (P))) to be obtained by taking the pushout along i and a. Dually, for a morphism $c : C' \longrightarrow C$, the pullback Ec along d and c defines the image of E under the map $\mathbb{W}(c, A)$. Moreover, we define on $\mathbb{W}(C,A)$ an addition (Baer's sum) by

$$E_1 + E_2 = \nabla_A \, (E_1 \oplus E_2) \, \Delta_C$$

where ∇_A and Δ_C are the codiagonal and diagonal maps, and $E_1 \oplus E_2$ is the direct sum of E_1 and E_2 in $\mathbb{W}(C \oplus C, A \oplus A)$.

Given a left weakly exact structure $\mathcal{D}$ on $\mathcal{A}$ and objects $A, C \in \mathcal{A}$, we define

$$\mathbb{D}_A(C) = \left\{ \overline{(i,d)} \quad | \quad A \xrightarrow{i} B \xrightarrow{d} C \ \text{ is a short exact sequence with } d \in \mathcal{D} \right\}$$

We also use the notation $\mathrm{Ext}^1_{\mathcal{D}}(C,A) = \mathbb{D}_A(C)$. Dually, we define $\mathrm{Ext}^1_{\mathcal{I}}(C,A) = \mathbb{I}_A(C)$ for a right weakly exact structure $\mathcal{I}$.

Lemma 4.2. Let $\mathcal{D}$ be a left weakly exact structure on $\mathcal{A}$. Then for each $A \in \mathcal{A}$, the construction in Definition 4.1 yields a functor $\mathbb{D}_A = \mathrm{Ext}^1_{\mathcal{D}}(-,A) : \mathcal{A}^{op} \to \mathrm{Set}$. Dually, for every object $C \in \mathcal{A}$, a right weakly exact structure $\mathcal{I}$ defines a functor $\mathbb{I}_C = \mathrm{Ext}^1_{\mathcal{I}}(C,-) : \mathcal{A} \to \mathrm{Set}$.

Proof. Adapt [M65, Chapter VII, Lemma 1.3 (i) and (ii)] to our context. □

Proposition 4.3. Let $\mathcal{W}$ be weakly exact structure on $\mathcal{A}$. Then the construction in Definition 4.1 yields an additive bifunctor

$$\mathbb{W} = \operatorname{Ext}^1_{\mathcal{W}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab};\ (C, A) \longmapsto \mathbb{W}(C, A).$$

Proof. This result can be obtained using the embedding of [GR92, Prop. 9.1] and the same techniques as used in [DRSS, Section 1.2]. Assume that $\mathcal{W}$ is a weakly exact structure on $\mathcal{A}$, and write $\mathcal{W} = (\mathcal{I}, \mathcal{D})$ with $\mathcal{I}$ a right weakly exact structure and $\mathcal{D}$ a left weakly exact structure, as defined in 3.13. The fact that $\mathbb{W}$ is a bifunctor then follows from Lemma 4.2 and [M65, Chapter VII, Lemma 1.3 (iii)]. □

Lemma 4.4. Let $\mathcal{V}$ and $\mathcal{W}$ be weakly exact structures on $\mathcal{A}$ with $\mathcal{V} \subseteq \mathcal{W}$. Then $\mathbb{V} = \operatorname{Ext}^1_{\mathcal{V}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Ab}$ is an additive sub-bifunctor of $\mathbb{W}$.

Proof. It is not complicated to check that $\mathbb{V}(C, A)$ is a subgroup of $\mathbb{W}(C, A)$. Moreover, multiplication by morphisms is given by pullback and pushout, and since $\mathcal{V}$ is stable under these operations, we have that $\mathbb{V}$ is an additive sub-bifunctor of $\mathbb{W}$. □

Remark 4.5. We consider the partial order on $\mathbf{BiFun}(\mathcal{A})$ given by

$$F \le F' \iff F(C, A) \le F'(C, A) \text{ for all } A, C \in \mathcal{A}$$

that is, $F(C, A)$ is a subgroup of $F'(C, A)$ for every pair of objects in $\mathcal{A}$. The construction in Definition 4.1 thus defines a map Φ from the weakly exact structures included in $\mathcal{E}_{max}$ on the additive category $\mathcal{A}$ to the $\mathcal{A} - \mathcal{A}-$bimodules:

$$\begin{aligned} \Phi : \mathbf{Wex}(\mathcal{A}) &\longrightarrow \mathbf{BiFun}(\mathcal{A}) \\ \mathcal{W} &\longmapsto \mathbb{W} = \operatorname{Ext}^1_{\mathcal{W}}(-,-). \end{aligned}$$

Lemma 4.4 shows that Φ is a morphism of posets. The elements in $\mathbf{Ex}(\mathcal{A})$ are sent under the map Φ to subfunctors of $\operatorname{Ext}^1_{\mathcal{A}}(-,-) = \mathbb{E}_{max}$ that enjoy an additional property, namely they give rise to a long exact sequence of functors:

Definition 4.6. ([BuHo61, DRSS]) An additive sub-bifunctor F of $\operatorname{Ext}^1_{\mathcal{A}}(-,-)$ is called *closed* if for any short exact sequence

$$E :\ A \xrightarrow{i} B \xrightarrow{d} C$$

whose class lies in $F(C, A)$ and any object X in $\mathcal{A}$, the sequences

$$0 \to \operatorname{Hom}(X, A) \to \operatorname{Hom}(X, B) \to \operatorname{Hom}(X, C) \to F(X, A) \to F(X, B) \to F(X, C)$$

and

$$0 \to \operatorname{Hom}(C, X) \to \operatorname{Hom}(B, X) \to \operatorname{Hom}(A, X) \to F(C, X) \to F(B, X) \to F(A, X)$$

are exact in the category of abelian groups. As noted in [BuHo61], the above sequences are always exact in all positions except at $F(X, B)$, respectively $F(B, X)$, thus one could equivalently say F is closed if the functors $F(X, -)$ and $F(-, X)$ are *middle-exact,* or using the terminology of [Rou, 4.1.1], *(co-)homological.*

Proposition 4.7. [DRSS, Prop 1.4] Let $\mathcal{E}$ be an exact structure on $\mathcal{A}$. Then the bifunctor $\Phi(\mathcal{E})$ is closed.

4.2. **From sub-bifunctors of $\mathbf{Ext}^1_{\mathcal{A}}$ to weakly exact structures.** We defined in the previous section a map

$$\Phi : \mathbf{Wex}(\mathcal{A}) \longrightarrow \mathbf{BiFun}(\mathcal{A}).$$

Our aim of this section is to construct a partial inverse function Ψ, so we start by this construction on the interval of weakly exact structures included in $\mathcal{E}_{max}$, denoted

$$\mathbf{Wex}(\mathcal{E}_{max}) := [\mathcal{E}_{min}, \mathcal{E}_{max}] \subseteq \mathbf{Wex}(\mathcal{A}).$$

Likewise, we write $\mathbf{BiFun}(\mathbb{E}_{max})$ for the class of sub-objects of $\mathbb{E}_{max}$ in $\mathbf{BiFun}(\mathcal{A})$. Formulated in terms of posets, one can say

$$\mathbf{BiFun}(\mathbb{E}_{max}) := [\mathbb{E}_{min}, \mathbb{E}_{max}] \subseteq \mathbf{BiFun}(\mathcal{A})$$

is the interval of all additive bifunctors between the minimum and the maximum exact structure on $\mathcal{A}$. Note that for a weakly idempotent complete category $\mathcal{A}$, or more generally under the conditions of Corollary 3.17, we have that $\mathcal{E}_{max}$ is the maximal weakly exact structure on $\mathcal{A}$, therefore $\mathbf{Wex}(\mathcal{E}_{max}) = \mathbf{Wex}(\mathcal{A})$.

To define a map Ψ on $\mathbf{BiFun}(\mathbb{E}_{max})$, we use the notion of $F-$exact pairs given in the following definition:

Definition 4.8. [BuHo61, DRSS] Let $F : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$ be an additive sub-bifunctor of $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$. Define a class $\mathcal{W}_F$ of short exact sequences by

$$\mathcal{W}_F := \{ A \xrightarrow{i} B \xrightarrow{d} C \text{ in } \mathcal{A} \mid \overline{(i,d)} \in F(C,A)\}.$$

The short exact sequences (i,d) in $\mathcal{W}_F$ are called F*-exact pairs.*

Proposition 4.9. ([BuHo61, DRSS]) The construction in Definition 4.8 yields a map

$$\begin{aligned} \Psi : \mathbf{BiFun}(\mathbb{E}_{max}) &\longrightarrow \mathbf{Wex}(\mathcal{E}_{max}) \\ F &\longmapsto \mathcal{W}_F. \end{aligned}$$

Moreover, the functions Φ and Ψ induce mutually inverse poset isomorphisms

$$\begin{array}{ccc} \mathbf{Wex}(\mathcal{E}_{max}) & \longleftrightarrow & \mathbf{BiFun}(\mathbb{E}_{max}) \\ \cup & & \cup \\ \mathbf{Ex}(\mathcal{A}) & \longleftrightarrow & \mathbf{CBiFun}(\mathcal{A}) \end{array}$$

where $\mathbf{CBiFun}(\mathcal{A})$ denotes the subclass of closed sub-bifunctors of $\mathrm{Ext}^1_{\mathcal{A}}$.

Proof. These results are mostly covered in [DRSS], some going back to [BuHo61]. $\square$

4.3. **Example.** We reconsider here Example 3.2 in light of the bijection from the last proposition: Let $\mathcal{A} = \operatorname{rep} Q$ be the category of representations of the quiver

$$Q: \qquad 1 \longrightarrow 2 \longrightarrow 3$$

The Auslander-Reiten quiver of $\mathcal{A}$ is as follows:

$$\begin{tikzcd}
& & P_1 \arrow[dr] & & \\
& P_2 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & I_2 \arrow[dr] & \\
P_3 \arrow[ur] \arrow[rr, dotted, no head] & & S_2 \arrow[ur] \arrow[rr, dotted, no head] & & S_1
\end{tikzcd}$$

There are (up to equivalence) exactly five non-split exact sequences with indecomposable end terms, where the first three are the Auslander-Reiten sequences:

$$(\alpha) \qquad 0 \longrightarrow P_3 \xrightarrow{a} P_2 \xrightarrow{c} S_2 \longrightarrow 0$$

$$(\beta) \qquad 0 \longrightarrow S_2 \xrightarrow{e} I_2 \xrightarrow{f} S_1 \longrightarrow 0$$

$$(\gamma) \qquad 0 \longrightarrow P_2 \longrightarrow P_1 \oplus S_2 \longrightarrow I_2 \longrightarrow 0$$

$$(\delta) \qquad 0 \longrightarrow P_3 \longrightarrow P_1 \xrightarrow{d} I_2 \longrightarrow 0$$

$$(\epsilon) \qquad 0 \longrightarrow P_2 \xrightarrow{b} P_1 \longrightarrow S_1 \longrightarrow 0$$

Up to isomorphism, an additive functor is uniquely determined by its values on indecomposable objects. To study additive sub-bifunctors of $\operatorname{Ext}^1_{\mathcal{A}}$ it is therefore sufficient to examine the bimodule structure on the vector space generated by these five non-split exact sequences with indecomposable end-terms. It is depicted in the following diagram, which indicates the multiplication rules $\delta e = \alpha, a\delta = \gamma, \epsilon f = \gamma, c\epsilon = \beta$ (see Definition 3.1) :

$$\begin{tikzcd}
& \epsilon \arrow[dl, "c"'] \arrow[dr, "f"] & & \delta \arrow[dl, "a"'] \arrow[dr, "e"] & \\
\beta & & \gamma & & \alpha
\end{tikzcd}$$

From there it is easy to see that $\operatorname{Ext}^1_{\mathcal{A}}$ admits 13 submodules (including the zero submodule and itself), and the submodule lattice is given in Figure 4.3, indicating each submodule by a set of generators. The eight closed submodules, corresponding to the eight exact structures on $\mathcal{A}$, are indicated in blue. Note that the submodule generated by the set of all Auslander–Reiten sequences $\{\alpha, \beta, \gamma\}$ corresponds to the Auslander–Reiten phantom morphisms studied in [FGHT13].

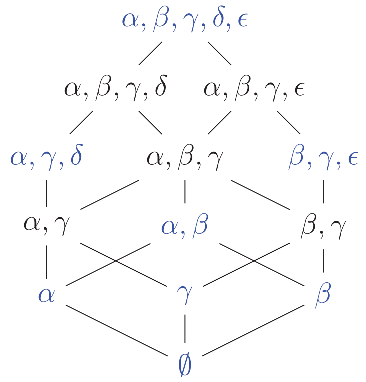

Figure 2: Subbimodules of $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$

4.4. **Weakly exact structures as bimodules.** In this part, we use the bifunctors, that we associated to weakly exact structures, to obtain bimodules over the Auslander algebra.

Definition 4.10. Let $\mathcal{A}$ be an additively finite, Hom-finite Krull-Schmidt category with indecomposables $X_1, \dots, X_n$ and denote by $A = \mathrm{End}(X)$ with $X = X_1 \oplus \cdots \oplus X_n$ its Auslander algebra. The Krull-Schmidt property implies that the additive category $\mathcal{A}$ is weakly idempotent complete, thus as discussed in Section 3.5, we know that the maximum weakly exact structure coincides with the maximum weakly exact structure formed by the stable short exact sequences. The corresponding bifunctor $\mathbb{E}_{max}$, evaluated at the object X yields a bimodule

$$\mathrm{B} = \mathbb{E}_{max}(X, X)$$

over the Auslander algebra A.

Let $\mathcal{W}$ be a weakly exact structure on $\mathcal{A}$, and consider its associated bifunctor $\mathbb{W} = \mathrm{Ext}^1_{\mathcal{W}}(-,-)$. We showed in Proposition 4.3 that the abelian group $B_{\mathcal{W}} = \mathbb{W}(X, X)$ forms a bimodule over the Auslander algebra B, and by Proposition 4.9, we obtain that $\mathrm{B}_{\mathcal{W}}$ is an $A-A-$subbimodule of B.

We denote by $\mathbf{Bim}(\mathrm{B})$ the class of all sub-bimodules of ${}_A\mathrm{B}_A$; it forms a poset $(\mathbf{Bim}(\mathrm{B}), \subseteq)$ with inclusion as order relation.

Example 4.11. In the example studied in Section 4.3, the Auslander algebra A is the algebra whose quiver is the Auslander-Reiten quiver with mesh relations, and the $A-A-$bimodule $\mathrm{B} = \mathbb{E}_{max}(X, X)$ is the Ext$-$bimodule on A, a five-dimensional bimodule with basis given by the elements $\alpha, \beta, \gamma, \delta, \epsilon$. The Figure 4.3 describes the bimodule lattice $(\mathbf{Bim}(\mathrm{B}), \subseteq)$ in this example.

5. Weakly extriangulated structures

Extriangulated structures [NP19] (or, equivalently, 1$-$exangulated structures [HLN]) generalize both exact and triangulated categories. In this Section, we generalise these categories, by defining their weak versions.

We recall the definition of $1-$exangulated categories following [HLN]:

Definition 5.1. Let $\mathbb{E} : \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Ab}$ be an additive bifunctor. Given a pair of objects $A, C \in \mathcal{A}$, we call an element $\delta \in \mathbb{E}(C, A)$ an $\mathbb{E}-$extension. When we want to emphasize A and C, we also write ${}_A\delta_C$.

Since $\mathbb{E}$ is a bifunctor, each morphism $a \in \mathrm{Hom}(A, A')$ induces the extension $a_*(\delta) := \mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A')$. Similarly, each morphism $c \in \mathrm{Hom}(C', C)$ induces the extension $c^*(\delta) := \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$.

Moreover, we have $\mathbb{E}(c, a)(\delta) = c^* a_*(\delta) = a_* c^*(\delta)$.

By the Yoneda lemma, each extension ${}_A\delta_C$ induces a pair of natural transformations

$$\delta_\sharp : \mathrm{Hom}(-, C) \to \mathbb{E}(-, A) \quad \text{and} \quad \delta^\sharp : \mathrm{Hom}(A, -) \to \mathbb{E}(C, -).$$

Namely, for each $X \in \mathcal{A}$, we have

$$(\delta_\sharp)_X : \mathrm{Hom}(X, C) \to \mathbb{E}(X, A), \quad c \mapsto c^*(\delta);$$
$$(\delta^\sharp)_X : \mathrm{Hom}(A, X) \to \mathbb{E}(C, X), \quad a \mapsto a_*(\delta).$$

Definition 5.2. A morphism of extensions ${}_A\delta_C \to {}_B\rho_D$ is a pair of morphisms $(a, c) \in \mathrm{Hom}(A, B) \times \mathrm{Hom}(C, D)$ such that $a_*(\delta) = c^*(\rho)$.

Definition 5.3. A *weak cokernel* of a morphism $f : A \to B$ in $\mathcal{A}$ is a morphism $g : B \to C$ such that for all $X \in \mathcal{A}$, the induced sequence of abelian groups

$$\mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$$

is exact, i.e. the sequence of functors

$$\mathrm{Hom}(C, -) \to \mathrm{Hom}(B, -) \to \mathrm{Hom}(A, -)$$

is exact. Equivalently, g is a weak cokernel of f if $g \circ f = 0$ and for each morphism $h : B \to X$ such that $h \circ f = 0$, there exists a (not necessarily unique) morphism $l : C \to X$ such that $h = l \circ g$. Weak kernel is a weak cokernel in $\mathcal{A}^{op}$.

Note that weak (co)kernels satisfy the same factorization properties as usual (co)kernels, but without requiring uniqueness. Clearly, a weak (co)kernel g of f is a (co)kernel of f if and only if g is a monomorphsim (resp. an epimorphism).

Definition 5.4. We call a pair of composable morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

a *weak kernel-cokernel pair* if f is a weak kernel of g and g is a weak cokernel of f.

By definition, in each weak kernel-cokernel pair as above the composition $g \circ f$ is 0, so the pair can be understood as an element of the category $\mathcal{C}^{[0,2]}(\mathcal{A}) \hookrightarrow \mathcal{C}(\mathcal{A})$ of complexes over $\mathcal{A}$ concentrated in the degrees $0, 1$ and 2.

Let $\mathcal{C}_w(\mathcal{A})$ be the full subcategory of $\mathcal{C}^{[0,2]}(\mathcal{A})$ with objects being weak kernel-cokernel pairs.

Consider morphisms of complexes in $\mathcal{C}_w(\mathcal{A})$

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle 1_A}\Big\| & & {\scriptstyle b}\Big\downarrow & & {\scriptstyle 1_C}\Big\| \\ A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C \end{array} \tag{1}$$

with leftmost and rightmost vertical morphisms being identities.

Lemma 5.5. For a diagram of the form (1), the following are equivalent:

- The morphism $h^\bullet = (1_A, b, 1_C)$ is an isomorphism in $\mathcal{C}_w(\mathcal{A})$;
- The morphism b is an isomorphism;
- The morphism $h^\bullet$ is a homotopy equivalence in $\mathcal{C}^{[0,2]}(\mathcal{A})$.

Here by homotopy equivalence in $\mathcal{C}^{[0,2]}(\mathcal{A})$ we mean that there exists a morphism $k^\bullet$ in $\mathcal{C}^{[0,2]}(\mathcal{A})$ and morphisms

$$\phi_1 : B \to A,\ \phi_2 : C \to B,\ \psi_1 : B' \to A,\ \psi_2 : C \to B'$$

such that the pair (ϕ_1, ϕ_2) yields a chain homotopy $k^\bullet \circ h^\bullet \sim 1$ and the pair (ϕ_1, ϕ_2) yields a chain homotopy $h^\bullet \circ k^\bullet \sim 1$.

Proof. This is a reformulation of [HLN, Lemma 4.1], see also [HLN, Claim 2.8]. □

Morphisms $h^\bullet = (1_A, b, 1_C)$ satisfying either of conditions in Lemma 5.5 define an equivalence relation on objects in $\mathcal{C}_w(\mathcal{A})$. We denote by $[A \xrightarrow{f} B \xrightarrow{g} C]$ the equivalence class of the complex $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}_w(\mathcal{A})$ under this equivalence.

Definition 5.6. (cf. [HLN, Definition 2.22]) Let $\mathfrak{s}$ be a correspondence which associates an equivalence class

$$\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$$

in $\mathcal{C}_{(\mathcal{A})}$ to each extension $\delta = {}_A\delta_C$. Such $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it satisfies the following condition for any $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$ and any $\mathfrak{s}(\rho) = [A' \xrightarrow{f'} B' \xrightarrow{g'} C']$:

(R0) For any morphism of extensions $(a, c) : \delta \to \rho$, there exists a morphism $b : B \to B'$ such that $h^\bullet = (a, b, c)$ is a morphism in $\mathcal{C}^{[0,2]}(\mathcal{A})$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal, "1_A"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "g"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal, "1_C"] \\
A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C.
\end{tikzcd}$$

Such $h^\bullet$ is called a *lift* of (a, c).

We say that $[A \xrightarrow{f} B \xrightarrow{g} C]$ *realizes* δ whenever we have $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$.

Each weak kernel-cokernel pair $A \xrightarrow{f} B \xrightarrow{g} C$ realizing an extension δ induces a pair of sequences of functors

$$\mathrm{Hom}(C, -) \to \mathrm{Hom}(B, -) \to \mathrm{Hom}(A, -) \to \mathbb{E}(C, -); \tag{2}$$

$$\mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B) \to \mathrm{Hom}(-, C) \to \mathbb{E}(-, A). \tag{3}$$

Definition 5.7. (cf. [HLN, Definition 2.22])
A realization $\mathfrak{s}$ is called *exact* if the following two conditions are satisfied:

(R1) For each extension δ, for each $A \xrightarrow{f} B \xrightarrow{g} C$ realizing δ, both sequences (2) are exact (i.e. exact when applied to each object in $\mathcal{A}$);
(R2) For each object $A \in \mathcal{A}$, we have

$$\mathfrak{s}({}_A0_0) = [A \xrightarrow{1_A} A \to 0], \quad \mathfrak{s}({}_00_A) = [0 \to A \xrightarrow{1_A} A].$$

Remark 5.8. Note that since we require realizations to be given by weak kernel-cokernel pairs, sequences (2) are automatically exact at $\mathrm{Hom}(B, -)$, resp. at $\mathrm{Hom}(-, B)$. In other words, condition (R1) concerns only exactness at $\mathrm{Hom}(A, -)$, resp. at $\mathrm{Hom}(-, C)$.

Remark 5.9. By [HLN, Proposition 2.16], condition (R1) does not depend on the choice of a representative in the equivalence class $\mathfrak{s}(\delta)$.

Definition 5.10. ([HLN, Definition 2.23], [NP19, Definition 2.15, Definition 2.19]) Let $\mathfrak{s}$ be an exact realization of $\mathbb{E}$. Pairs $\delta, \mathfrak{s}(\delta)$ are called (*distinguished*) $\mathbb{E}-$triangles. If a complex

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a representative in $\mathfrak{s}(\delta)$ for some δ, it is called a *conflation*. In this case, the morphism f is called an *inflation* and the morphism g is called a deflation.

Lemma 5.11. ([HLN, Proposition 3.2]) The class of conflations and the class of $\mathbb{E}-$triangles are both closed under direct sums and direct summands.

Since we consider weak kernel-cokernel pairs as complexes, we can consider mapping cones and cocones of morphisms between them. We use the minor modification of the usual definition that was considered in [HLN] and applies only for certain morphisms.

Definition 5.12. [HLN, Definition 2.27] Let $f^\bullet = (1_A, b, c)$ be a morphism in $\mathcal{C}^{[0,1]}(\mathcal{A})$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal, "1_A"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[d, "c"'] \\ A \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'. \end{tikzcd}$$

Its *mapping cone* $M_f^\bullet$ is defined to be the complex

$$B \xrightarrow{\begin{bmatrix} -g \\ b \end{bmatrix}} C \oplus B' \xrightarrow{\begin{bmatrix} c & g' \end{bmatrix}} C'.$$

In other words, this is the usual mapping cone of the morphism of complexes

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[d, "c"'] \\ B' \arrow[r, "g'"] & C'. \end{tikzcd}$$

The *mapping cocones* (cylinder) of morphisms of the form $(a, b, 1_C)$ are defined dually.

Definition 5.13. ([HLN, Definition 2.32 for $n = 1$]) A 1-exangulated category is a triplet $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ of an additive category $\mathcal{A}$, additive bifunctor $\mathbb{E} : \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Ab}$, and its exact realization $\mathfrak{s}$, satisfying the following conditions.

(EA1) The composition of two inflations is an inflation. Dually, the composition of two deflations is a deflation.

(EA2) For each $\rho \in \mathbb{E}(C', A)$ and $c \in \mathrm{Hom}(C, C')$, for each pair of realizations $A \xrightarrow{f} B \xrightarrow{g} C$ of $c^*\rho$ and $A \xrightarrow{f'} B' \xrightarrow{g'} C'$ of ρ, the morphism $(1_A, c) : c^*\rho \to \rho$ admits a *good lift* $f^\bullet = (1_A, b, c)$, in the sense that $M_f^\bullet$ realizes $f_*\rho$.

(EA2)^{op} Dual of (EA2).

Proposition 5.14. ([HLN, Proposition 4.3]) A triplet $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ is a 1-exangulated category if and only if it is an extriangulated category as defined by Nakaoka and Palu [NP19].

This result motivates the following definition.

Definition 5.15. A *weakly extriangulated* (= *weakly* $1-$*exangulated*) structure on an additive category $\mathcal{A}$ is a pair $(\mathbb{W}, \mathfrak{s})$ of an additive bifunctor $\mathbb{W} : \mathcal{A}^{op} \times \mathcal{A} \to \mathrm{Ab}$ and its exact realization $\mathfrak{s}$ satisfying axioms (EA2) and (EA2)^{op}.

Let $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$ be a weakly extriangulated structure. Assume that $\mathbb{W}'$ is an additive sub-bifunctor of $\mathbb{W}$. Consider the restriction $\mathfrak{s}|_{\mathbb{W}'}$ of the realization $\mathfrak{s}$ on $\coprod_{c,a\in\mathcal{A}} \mathbb{W}'(c, a)$. The following immediately follows from the definitions. The case of $(\mathbb{W}, \mathfrak{s})$ extriangulated was considered in [HLN, Claim 3.8].

Lemma 5.16. $(\mathbb{W}', \mathfrak{s}|_{\mathbb{W}'})$ is a weakly extraingulated struture on $\mathcal{A}$.

We say that $(\mathbb{W}', \mathfrak{s}|_{\mathbb{W}'})$ is a *weakly extriagnulated substructure* of $(\mathbb{W}, \mathfrak{s})$.

Lemma 5.17. A weakly exact structure $\mathcal{W}$ on $\mathcal{A}$ defines a weakly extriangulated structure $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$.

Proof. Using Lemma 3.7, all the arguments from [NP19, Example 2.13], except for those concerning (ET4) and (ET4)$^{\mathrm{op}}$, apply here word for word. That means that a weakly exact structure defines a pair of a bifunctor and its exact realization. Axioms (EA2) and (EA2)^{op} follow directly from axioms (E2) and (E2) combined with Lemma 3.5 and its dual. □

We can also characterize weakly exact structures among weakly extriangulated ones.

Lemma 5.18. (cf. [NP19, Corollary 3.18]) Let $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$ be a weakly extriangulated category, in which each inflation is monomorphic, and each deflation is epimorphic. If we let $\mathcal{W}$ be the class of conflations given by the $\mathbb{W}-$triangles, then $(\mathcal{A}, \mathcal{W})$ is a weakly exact category.

Proof. If an inflation in a conflation is monomorphic, it is not just a weak kernel of the deflation, but *the* actual kernel. Similarly, if a deflation is epimorphic, it is the cokernel of an inflation. Therefore, if each inflation is monomorphic, and each deflation is epimorphic, all conflations are kernel-cokernel pairs. From the exactness of the realization, it follows that the class of conflations is closed under direct sums and axioms (E0) and (E0)^{op} are satisfied. Axioms (EA2) and (EA2)^{op} imply the axioms (E2) and (E2)^{op} by Lemma 3.5 and its dual. □

Breaz and Modoi [BM15] introduced the notions of *almost exact structures* on full extension-closed subcategories $\mathcal{A}$ of triangulated categories $\mathcal{T}$ in terms of *proper classes of triangles* (generalizing work of Beligiannis [Bel00]) and of *phantom* $\mathcal{A}$ *ideals* of morphisms in $\mathcal{T}$. They found [BM15, Proposition 2.2.4] a bijection between almost exact structures on $\mathcal{A}$ and phantom $\mathcal{A}-$ideals.

Lemma 5.19. Each pair of a phantom $\mathcal{A}-$ideal in $\mathcal{T}$ and the corresponding proper class of triangles yields a weakly extriangulated structure on $\mathcal{A}$.

Proof. This follows from [BM15, Remark 2.2.3 (ii), Proposition 2.2.4], the fact that $\mathcal{A}$ is extriangulated with the structure induced by that of $\mathcal{T}$, and Lemma 5.16. □

6. Defects and topologizing subcategories

In this section, we extend the notion of contravariant defects to the setting of weakly extriangulated categories. These categories were used in [Bu01, En18, En20, FG20] to classify exact structures on an additive category and, more generally, extriangulated substructures of an extriangulated structure. We show that their results extend to our framework.

Definition 6.1. Let $\mathcal{A}$ be an essentially small additive category. Contravariant additive functors $\mathcal{A}^{op} \to \mathrm{Ab}$ to the category of abelian groups are called *right* $\mathcal{A}-$*modules*. They form an abelian category $\mathrm{Mod}\,\mathcal{A}$. Dually, *left* $\mathcal{A}-$*modules* are covariant additive functors to abelian groups, they form an abelian category that can be seen as $\mathrm{Mod}\,\mathcal{A}^{op}$.

These categories have enough projectives. Those are precisely the direct summands of direct sums of representable functors $\mathrm{Hom}(-,A) \in \mathrm{Mod}\,\mathcal{A}$, resp. $\mathrm{Hom}(A,-) \in \mathrm{Mod}\,\mathcal{A}^{op}$.

We will work with certain full subcategories of categories of $\mathcal{A}-$modules. First, we need to recall several classical definitions:

Definition 6.2. An $\mathcal{A}$-module M is called *finitely generated* if admits an epimorphism $\mathrm{Hom}(-,X) \twoheadrightarrow M$ from a representable functor. It is moreover *finitely presented* if it is a cokernel of a morphism of representable functors. A module is called *coherent* if it is finitely presented and each of its finitely generated submodule is also finitely presented. Note that every finitely generated submodule of a coherent module is automatically coherent.

By definition, we have a chain of embeddings of full additive categories

$$\mathbf{coh}(\mathcal{A}) \hookrightarrow \mathbf{fp}(\mathcal{A}) \hookrightarrow \mathbf{fg}(\mathcal{A}) \hookrightarrow \mathrm{Mod}\,\mathcal{A},$$

where the first three categories are the categories of coherent, finitely presented and finitely generated right $\mathcal{A}-$modules, respectively.

The category of finitely presented modules $\mathbf{fp}(\mathcal{A})$ is known to be abelian if and only the category $\mathcal{A}$ has weak kernels. The category of coherent modules behaves better, as the following standard fact shows:

Proposition 6.3. ([He97, Proposition 1.5], see also [Fi16, Appendix B]) The category $\mathbf{coh}(\mathcal{A})$ is abelian and the canonical embedding $\mathbf{coh}(\mathcal{A}) \hookrightarrow \mathrm{Mod}\,\mathcal{A}$ is exact. In particular, $\mathbf{coh}(\mathcal{A})$ is closed under kernels, cokernels and extensions in $\mathrm{Mod}\,(\mathcal{A})$[1].

Two more important full subcategories of categories of modules over abelian categories has been studied thoroughly since 1950s and 1960s: the category of

[1]Full subcategories of abelian categories, which are closed under kernels, cokernels and extensions, are sometimes also called *wide* subcategories.

effaceable functors, studied already by Grothendieck [Gr57], and the category of defects introduced by Auslander [A66, A78, ARS]. These notions have been generalized to the setting of exact categories (see e.g. [Ke90, Fi16, En18]) and, by Ogawa [Og19] and Enomoto [En20], to that of extriangulated categories. Ogawa's definition uses only part of the axioms of extriangulated categories, and so we can formulate it in our broader context.

Let $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$ be a weakly extriangulated category.

Definition 6.4. We say that a module $F \in \operatorname{Mod} \mathcal{A}$ is *weakly effaceable with respect to* $(\mathbb{W}, \mathfrak{s})$ if the following condition is satisfied:

For any $Z \in \mathcal{A}$ and any $z \in F(Z)$, there exists a deflation $g : Y \twoheadrightarrow Z$ such that $F(g)(z) = 0$.

Definition 6.5. Given a conflation $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$, we define its *contravariant defect* to be the cokernel of $\operatorname{Hom}(-, g) : \operatorname{Hom}(-, Y) \to \operatorname{Hom}(-, Z)$ in $\operatorname{Mod} \mathcal{A}$.

We denote by $\mathbf{Eff}\, \mathbb{W}$ the category of weakly effaceable functors and by $\mathbf{def}\, \mathbb{W}$ the full subcategory of right $\mathcal{A}-$modules isomorphic to defects of conflations. If $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$ corresponded to a weakly exact structure $\mathcal{W}$ on $\mathcal{A}$, we also write $\mathbf{Eff}\, \mathcal{W} := \mathbf{Eff}\, \mathbb{W}$ and $\mathbf{def}\, \mathcal{W} := \mathbf{Eff}\, \mathcal{W}$.

For abelian categories endowed with maximal exact structures, the following two statements are standard, see e.g. [Gr57], resp. [ARS].

Lemma 6.6. The category $\mathbf{Eff}\, \mathbb{W}$ is closed under subquotients and finite direct sums in $\operatorname{Mod} \mathcal{A}$.

Proof. Let

$$0 \to F \overset{\mu}{\to} G \overset{\nu}{\to} H \to 0$$

be a short exact sequence in $\operatorname{Mod} \mathcal{A}$. Assume that G is weakly effaceable with respect to $(\mathbb{W}, \mathfrak{s})$. Let Z be an object of $\mathcal{A}$. Choose an element $z \in F(Z)$ and a deflation $f : P \to Z$ such that

$$0 = G(f) \circ \mu(Z)(z) = \mu(P) \circ F(f)(z).$$

Since μ is monic, $F(f)(z) = 0$. Thus, F is weakly effaceable with respect to $(\mathbb{W}, \mathfrak{s})$. So $\mathbf{Eff}\, \mathbb{W}$ is closed under subobjects. The rest is proved by similar straightforward diagram chasing. □

Lemma 6.7. The category $\mathbf{def}\, \mathbb{W}$ is closed under kernels and cokernels in $\operatorname{Mod} \mathcal{A}$.

Proof. The same argument as in [Og19, Lemma 2.6] applies here. A morphism of defects of two conflations gives rise to a morphism (a, c) of these conflations. Then the kernel is given by the defect of the mapping cone of any good lift of the morphism $(1, c)$ and the cokernel is given by the defect of the mapping cocone of any good lift of the morphism $(a, 1)$. □

The following notion was introduced by Rosenberg [Ros] in his works on noncommutative algebraic geometry and reconstruction of schemes.

Definition 6.8. A full subcategory of an abelian category is called *topologizing* if it is closed under subquotients and finite direct sums.

Proposition 6.9. Let $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$ be a weakly extriangulated category. We have

$$\mathbf{def}\,\mathbb{W} = \mathbf{Eff}\,\mathbb{W} \bigcap \mathbf{coh}(\mathcal{A})$$

and this category is topologizing.

Proof. The same argument as in the proof of [En20, Proposition 2.9] applies here. The only difference is that in our generality $\mathbf{Eff}\,\mathbb{W}$ is not closed under extensions in $\operatorname{Mod}\mathcal{A}$, but only under finite direct sums. □

For $\mathcal{A}-$modules, we have natural notions of subobjects, quotients and extensions: these are defined object-wise (for objects in $\mathcal{A}$).

Definition 6.10. We say that a subcategory of an arbitrary (not necessarily abelian) full subcategory $\mathcal{C}$ of $\mathbf{coh}(\mathcal{A})$ is *topologizing* if it is closed under subquotients (considered object-wise) and finite direct sums. Equivalently, it is topologizing if it is a full subcategory of $\mathcal{C}$ which is topologizing in $\mathbf{coh}(\mathcal{A})$.

Similarly, we say that a subcategory of $\mathcal{C}$ is *Serre* if it is topologizing and closed under extensions; equivalently, if it is a full subcategory of $\mathcal{C}$ and a Serre subcategory in $\mathbf{coh}(\mathcal{A})$.

Note that this definition ensures that a Serre subcategory of $\mathcal{C}$ is abelian.

Corollary 6.11. Let $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$ be a weakly extriangulated category and let $(\mathcal{A}, \mathbb{W}', \mathfrak{s}|_{\mathbb{W}'})$ be a weakly extriangulated substructure on $\mathcal{A}$. Then the category $\mathbf{def}\,\mathbb{W}'$ is a topologizing subcategory of $\mathbf{def}\,\mathbb{W}$.

Corollary 6.12. Let $\mathcal{W}'$ be a weakly exact substructure of a weakly exact structure $\mathcal{W}$. Then the category $\mathbf{def}\,\mathcal{W}'$ is a topologizing subcategory of $\mathbf{def}\,\mathcal{W}$.

7. Lattice structures

We study in this section lattice structures on the different posets introduced in the previous parts of this article.

7.1. Definitions.

We recall the following well known notions:

Definition 7.1. A poset P is called a *join-semilattice* if for every pair (p, q) of elements of P there exists a supremum $p \vee q$ (also called join). It is called a *meet-semilattice* if for every pair (p, q) of elements of P there exists an infimum $p \wedge q$ (also called meet). Finally, P is *lattice* if it is both a join-semilattice and a meet-semilattice. Equivalently, a lattice is a set P equipped with two binary operations $\vee$ and $\wedge : P \times P \to P$ satisfying the following axioms:

1. $\vee$ is associative and commutative,
2. $\wedge$ is associative and commutative,

(3) $\wedge$ and $\vee$ satisfy the following property:

$$m \vee (m \wedge n) = m = m \wedge (m \vee n) \text{ for all } m, n \in P.$$

A lattice is called *complete* if all its subsets have both a join and a meet, similar for semilattices. A *bounded* lattice is a lattice that has a greatest element (also called maximum) and a least element (also called minimum).

Remark 7.2. As a consequence of the axioms above we have the following property for lattices:

$$m \vee m = m \text{ and } m \wedge m = m \text{ for all } m \in P.$$

Definition 7.3. A lattice $(P, \leq, \wedge, \vee)$ is *modular* if the following property is satisfied for all $r, s, t \in P$ with $r \leq s$:

$$s \wedge (r \vee t) = r \vee (s \wedge t).$$

Definition 7.4. [Da02, 2.16, 2.17] Let P and Q be two lattices, then a function $f : P \to Q$ is a *morphism of lattices* if for all $m, n \in P$ one has:

$$f(m \vee n) = f(m) \vee f(n) \qquad \textit{and} \qquad f(m \wedge n) = f(m) \wedge f(n).$$

An *isomorphism of lattices* is a bijective morphism of lattices (in which case its inverse is also an isomorphism).

Definition 7.5. Let $(P, \leqslant)$ be a partially ordered set with a unique minimal element 0. An *atom* is an element $a \in P$ with $a > 0$ and such that $0 \leqslant x \leqslant a$ implies $x = 0$ or $x = a$. In other words, atoms are the elements that are directly above the minimal element.

7.2. **Lattices of right and left weakly exact structures.** In this subsection we study a lattice structure on the class of all right (or left) weakly exact structures. These results generalise the one obtained in [HR20, Proposition 8.4] on the complete lattice structure of the class of (strong) one-sided exact structures.

Definition 7.6. We denote by $\mathbf{LW}(\mathcal{A})$ (respectively $\mathbf{RW}(\mathcal{A})$) the class of all left (right) weakly exact structures on $\mathcal{A}$.

Lemma 7.7. Let $\{\mathcal{L}_i\}_{i\in\omega}$ ($\{\mathcal{R}_i\}_{i\in\omega}$) be a family of left (right) weakly exact structures on $\mathcal{A}$. Then the intersection $\cap_{i\in\omega}\mathcal{L}_i$ ($\cap_{i\in\omega}\mathcal{R}_i$) is also a left (right) weakly exact structure.

Proof. Same as Lemma 5.2 of [BHLR]. □

Proposition 7.8. Let $\mathcal{A}$ be an additive category. Then $\mathbf{LW}(\mathcal{A})$ and $\mathbf{RW}(\mathcal{A})$) are complete meet-semi lattices.

Proof. Let $\mathcal{L}$ and $\mathcal{L}'$ be two left weakly exact structures on $\mathcal{A}$. The partial order on $\mathbf{LW}(\mathcal{A})$ is given by containment. We define the *meet* given by $\mathcal{L} \wedge \mathcal{L}' = \mathcal{L} \cap \mathcal{L}'$. These operations define the structure of complete meet-semilattice on $LW(\mathcal{A})$ by Lemma 7.7. □

Remark 7.9. If there exists a unique maximal left weakly exact structure $\mathcal{L}_{max}$ on $\mathcal{A}$, then $\mathbf{LW}(\mathcal{A})$ is a complete lattice (similarly for $\mathbf{RW}(\mathcal{A})$). In this case, the *join* can be defined by the usual construction

$$\mathcal{L} \vee_L \mathcal{L}' = \cap\{\mathcal{L}'' \in \mathbf{LW}(\mathcal{A}) \mid \mathcal{L} \subseteq \mathcal{L}'', \mathcal{L}' \subseteq \mathcal{L}''\}.$$

The intersection in the definition of the join is well defined since the set includes $\mathcal{L}_{max}$ by assumption. These operations define a lattice structure on $\mathbf{LW}(\mathcal{A})$. Since the lattice has a minimal element $\mathcal{L}_{min}$, formed by all retractions, and a maximal element $\mathcal{L}_{max}$, it is a bounded lattice. Likewise, any interval in the poset $\mathbf{LW}(\mathcal{A})$ forms a complete bounded lattice.

Remark 7.10. The constructions in Section 3.4 can be reformulated in terms of the lattices studied in this section as follows: As stated in Proposition 3.4, there is a splicing function

$$s : \mathbf{Wex}(\mathcal{A}) \longrightarrow \mathbf{LW}(\mathcal{A}) \times \mathbf{RW}(\mathcal{A}),\ \mathcal{W} \longmapsto (\mathcal{L}_{\mathcal{W}}, \mathcal{R}_{\mathcal{W}})$$

where $\mathcal{L}_{\mathcal{W}} := \{\ d \mid (i,d) \in \mathcal{W}\}$ is the class of all $\mathcal{W}-$cokernels or $\mathcal{W}-$admissible deflations and $\mathcal{R}_{\mathcal{W}} := \{\ i \mid (i,d) \in \mathcal{W}\}$ is the class of all $\mathcal{W}-$kernels or $\mathcal{W}-$admissible inflations.

Moreover, Theorem 3.13 shows that there is a gluing function:

$$g : \mathbf{LW}(\mathcal{A}) \times \mathbf{RW}(\mathcal{A}) \longrightarrow \mathbf{Wex}(\mathcal{A}), (\mathcal{L}, \mathcal{R}) \longmapsto \mathcal{W}_{(\mathcal{L},\mathcal{R})}$$

where $\mathcal{W}_{(\mathcal{L},\mathcal{R})}$ is formed by all short exact sequences (i,d) in $\mathcal{A}$ with $i \in \mathcal{R}, d \in \mathcal{L}\}$.

The maps s and g are not bijective, but it seems interesting to study their properties.

7.3. **Lattice of weakly exact structures.**

7.3.1. *Lattice of exact structures revisited.* We know by [BHLR, Theorem 5.3] that the class of exact structures on an additive category $\mathbf{Ex}(\mathcal{A})$ forms a lattice. In order to study the properties of this lattice, we show that it is isomorphic to the lattice of closed additive sub-bifunctors of $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$ defined in Section 4.

Theorem 7.11. [BHLR, 5.1, 5.3, 5.4] Let $\mathcal{A}$ be an additive category. The poset $\mathbf{Ex}(\mathcal{A})$ of exact structures $\mathcal{E}$ on $\mathcal{A}$ forms a bounded complete lattice

$$(\mathbf{Ex}(\mathcal{A}), \subseteq, \wedge, \vee_E)$$

under the following operations:

(1) The partial order is given by containment $\mathcal{E}' \subseteq \mathcal{E}$
(2) The meet $\wedge$ is defined by $\mathcal{E} \wedge \mathcal{E}' = \mathcal{E} \cap \mathcal{E}'$
(3) the join $\vee_E$ is defined by

$$\mathcal{E} \vee_E \mathcal{E}' = \bigcap\{\mathcal{F} \in \mathbf{Ex}(\mathcal{A}) \mid \mathcal{E} \subseteq \mathcal{F}, \mathcal{E}' \subseteq \mathcal{F}\}.$$

7.3.2. *Lattice structure on the class of all weakly exact structures of a given additive category.*

Lemma 7.12. Let $\{\mathcal{W}_i\}_{i\in\omega}$ be a family of weakly exact structures on $\mathcal{A}$. Then the intersection $\cap_{i\in\omega}\mathcal{W}_i$ is also a weakly exact structure.

Proof. Same as Lemma 5.2 of [BHLR]. □

Theorem 7.13. Let $\mathcal{A}$ be an additive category and $\mathcal{E}_{max}$ the maximal exact structure on $\mathcal{A}$. Then the weakly exact structures that are included in $\mathcal{E}_{max}$ form a complete bounded lattice:

$$(\mathbf{Wex}(\mathcal{E}_{max}), \subseteq, \wedge, \vee_W)$$

Proof. It follows from Lemma 7.12 that $\mathbf{Wex}(\mathcal{A})$ forms a meet semi-lattice: $(\mathbf{Wex}(\mathcal{A}), \subseteq, \wedge)$ with order relation given by inclusion and meet operation given by inclusion. Moreover, the weakly exact structures that are included in $\mathcal{E}_{max}$ form a complete bounded lattice $(\mathbf{Wex}(\mathcal{E}_{max}), \subseteq, \wedge, \vee_W)$ where the *join* $\vee_W$ is defined by

$$\mathcal{W} \vee_W \mathcal{W}' = \cap\{\mathcal{V} \in \mathbf{Wex}(\mathcal{A}) \mid \mathcal{W} \subseteq \mathcal{V}, \mathcal{W}' \subseteq \mathcal{V}\}$$

This join is well-defined for $\mathbf{Wex}(\mathcal{E}_{max})$ since the set includes $\mathcal{E}_{max}$ by assumption. Since the lattice $\mathbf{Wex}(\mathcal{E}_{max})$ has a minimal element $\mathcal{E}_{min}$ and a maximal element $\mathcal{E}_{max}$, it is a bounded lattice. □

Remark 7.14. While the partial order and the meet coincide for $\mathbf{Ex}(\mathcal{A})$ and $\mathbf{Wex}(\mathcal{A})$, the join $\vee_E$ is different from the join for weakly exact structures since we intersect over a *smaller set*, making the join *larger* when both are viewed in the poset $\mathbf{Wex}(\mathcal{E}_{max})$:

$$\mathcal{E} \vee_W \mathcal{E}' \quad \leq \quad \mathcal{E} \vee_E \mathcal{E}'$$

for all $\mathcal{E}, \mathcal{E}' \in \mathbf{Ex}(\mathcal{A})$. In fact, in the example from Section 4.3, if we consider the exact structures $\mathcal{E} = \langle\alpha\rangle$ and $\mathcal{E}' = \langle\gamma\rangle$, then $\mathcal{E} \vee_W \mathcal{E}' = \langle\alpha, \gamma\rangle$ which is stricly smaller than $\mathcal{E} \vee_E \mathcal{E}' = \langle\alpha, \gamma, \delta\rangle$. This shows that $\mathbf{Ex}(\mathcal{A})$ is a meet-subsemilattice of $\mathbf{Wex}(\mathcal{E}_{max})$, but it is *not* a sublattice in general.

We now describe the join of two weakly exact structures in a more constructive way, motivated by the sum of bifunctors:

Definition 7.15. Let $\mathcal{W}_1, \mathcal{W}_2 \in \mathbf{Wex}(\mathcal{E}_{max})$ be two weakly exact structures contained in $\mathcal{E}_{max}$. Then, $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ is defined as $\mathcal{W} := \bigcup_{A,C\in\mathcal{A}} \mathcal{W}(C, A)$ where

$$\mathcal{W}(C, A) := \{\eta_1 + \eta_2 \mid \eta_1 \in \mathcal{W}_1(C, A),\ \eta_2 \in \mathcal{W}_2(C, A)\}$$

with $W_k(C, A) := \{\eta : A \xrightarrow{i} B \xrightarrow{d} C \mid \eta \in \mathcal{W}_k\}$ for $k = 1, 2$. Here, for $\eta_1 \in \mathcal{W}_1(C, A)$ and $\eta_2 \in \mathcal{W}_2(C, A)$, the sum $\eta_1 + \eta_2 := \nabla_A(\eta_1 \oplus \eta_2)\Delta_C$ is the Baer sum for short exact sequences. Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are included in $\mathcal{E}_{max}$ and the Baer sum in well defined in $\mathcal{E}_{max}$, we have $\mathcal{W} \subseteq \mathcal{E}_{max}$.

Proposition 7.16. Let $\mathcal{W}_1, \mathcal{W}_2$ be two weakly exact structures contained in $\mathcal{E}_{max}$. Then

(a) $\mathcal{W}_1 + \mathcal{W}_2$ is weakly exact
(b) $\mathcal{W}_1 + \mathcal{W}_2$ is the join $\mathcal{W}_1 \vee_W \mathcal{W}_2$ in the lattice $\mathbf{Wex}(\mathcal{E}_{max})$.

Proof. (a) It is not complicated to show that $\mathcal{W}_1 + \mathcal{W}_2$ satisfies $(E0)$, $(E2)$ and their dual $(E0, E2)^{op}$. Moreover, suppose that $\alpha : \ A \xrightarrow{i} B \xrightarrow{d} C \ \in \mathcal{W}(C, A)$ and $\beta : \ D \xrightarrow{j} E \xrightarrow{e} F \ \in \mathcal{W}(F, D)$. Then there exist $\alpha_1 \in \mathcal{W}_1(C, A)$, $\alpha_2 \in \mathcal{W}_2(C, A)$, $\beta_1 \in \mathcal{W}_1(F, D)$ and $\beta_2 \in \mathcal{W}_2(F, D)$ such that $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$, hence

$$\alpha \oplus \beta = (\alpha_1 + \alpha_2) \oplus (\beta_1 + \beta_2) = (\nabla_A(\alpha_1 + \alpha_2)\Delta_C) \oplus (\nabla_D(\beta_1 + \beta_2)\Delta_F).$$

Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are closed under direct sums, we get $\alpha_1 \oplus \beta_1 \in \mathcal{W}_1(C \oplus F, A \oplus D)$ and $\alpha_2 \oplus \beta_2 \in \mathcal{W}_2(C \oplus F, A \oplus D)$, so $(\alpha_1 \oplus \beta_1) + (\alpha_2 \oplus \beta_2) \in \mathcal{W}(C \oplus F, A \oplus D)$. We have $(\alpha_1 \oplus \beta_1) + (\alpha_2 \oplus \beta_2) = \nabla_{A \oplus D}((\alpha_1 \oplus \beta_1) \oplus (\alpha_2 \oplus \beta_2))\Delta_{C \oplus F} = \nabla_{A \oplus D}((\alpha_1 + \alpha_2) \oplus (\beta_1 + \beta_2))\Delta_{C \oplus F}$. Note that the direct sum of the diagrams for $(\nabla_A(\alpha_1 + \alpha_2)\Delta_C)$ and $(\nabla_D(\beta_1 + \beta_2)\Delta_F)$ is the diagram for $\nabla_{A \oplus D}((\alpha_1 + \alpha_2) \oplus (\beta_1 + \beta_2))\Delta_{C \oplus F}$. This means that $\alpha \oplus \beta = (\alpha_1 \oplus \beta_1) + (\alpha_2 \oplus \beta_2) \in \mathcal{W}(C \oplus F, A \oplus D) \subseteq \mathcal{W}$. Therefore $\mathcal{W}$ is closed under direct sums and it is a weakly exact structure.

(b), recall that the join $\mathcal{W}_1 \vee_W \mathcal{W}_2$ is the smallest (by inclusion) weakly exact structure on $\mathcal{A}$ containing both $\mathcal{W}_1$ and $\mathcal{W}_2$. We have that $\mathcal{W}_1 \subset \mathcal{W}_1 + \mathcal{W}_2$ since $\eta_1 = \eta_1 + 0 \in \mathcal{W}_1 + \mathcal{W}_2$ for any $\eta_1 \in \mathcal{W}_1$. Likewise for $\mathcal{W}_2$, so $\mathcal{W}_1 + \mathcal{W}_2$ contains both $\mathcal{W}_1$ and $\mathcal{W}_2$, hence by definition of the join, $\mathcal{W}_1 \vee_W \mathcal{W}_2 \subseteq \mathcal{W}_1 + \mathcal{W}_2$.
To show the converse inclusion, let $\mathcal{W}$ be any weakly exact structure containing both $\mathcal{W}_1$ and $\mathcal{W}_2$. Since $\mathcal{W}$ satisfies the direct sum property (S), we have $\eta_1 \oplus \eta_2 \in \mathcal{W}$ for all $\eta_1 \in \mathcal{W}_1, \eta_2 \in \mathcal{W}_2$. By definition of Baer sum and property (E2) and (E2)^{op} for $\mathcal{W}$ we have $\eta_1 + \eta_2 \in \mathcal{W}$. This shows $\mathcal{W}_1 + \mathcal{W}_2 \subset \mathcal{W}$ for *all* $\mathcal{W}$ containing both $\mathcal{W}_1$ and $\mathcal{W}_2$, so this also holds for the smallest one (their intersection) : $\mathcal{W}_1 + \mathcal{W}_2 \subseteq \mathcal{W}_1 \vee_W \mathcal{W}_2$. □

Proposition 7.17. Let α be an Auslander-Reiten sequence in $\mathcal{A}$, and denote by $\mathcal{E}_\alpha = \{X \oplus Y \mid X \in \mathcal{E}_{min}, Y \in add(\alpha)\}$ the (weakly) exact structure generated by α. Then $\mathcal{E}_\alpha$ is an atom of both lattices $(\mathbf{Ex}(\mathcal{A}), \subseteq, \wedge, \vee_E)$ and $(\mathbf{Wex}(\mathcal{A}), \subseteq, \wedge, \vee_W)$.

Proof. This property amounts to showing that the Auslander-Reiten sequence lies in the socle of the bifunctor $\mathrm{Ext}^1_{\mathcal{A}}(-,-)$, that is, multiplication with morphisms does not generate any new non-split sequences. This is a well-known property of almost split sequences. □

7.4. **Lattice of additive sub-bifunctors of $\mathrm{Ext}^1_{\mathcal{A}}$.** In Section 4, we discussed additive sub-bifunctors of $\mathrm{Ext}^1_{\mathcal{A}} := \mathbb{E}_{max} = \mathrm{Ext}^1_{\mathcal{E}_{max}}$ and closed additive sub-bifunctors, and we denote these classes respectively by $\mathbf{BiFun}(\mathbb{E}_{max})$ and $\mathbf{CBiFun}(\mathcal{A})$. In this section, we construct lattice structures of both classes.

Theorem 7.18. The additive sub-bifunctors of $\mathbb{E}_{max}$ form a lattice

$$(\mathbf{BiFun}(\mathbb{E}_{max}), \leq, \wedge, \vee_{bf}).$$

Proof. For $F, F' \in \mathbf{BiFun}(\mathbb{E}_{max})$, we write $F \leq F'$ if F is a sub-bifunctor of F'. The meet of F and F' is given by the sub-bifunctor $F \wedge F'$ of $\mathbb{E}_{max}$ satisfying

$$(F \wedge F')(C, A) = F(C, A) \cap F'(C, A) \text{ for all } A, C \in \mathcal{A}.$$

The join is given by the sub-bifunctor $F + F' = F \vee_{bf} F'$ of $\mathbb{E}_{max}$ satisfying

$$(F \vee_{bf} F')(C, A) = F(C, A) + F'(C, A) \text{ for all } A, C \in \mathcal{A},$$

where the sum is the sum of abelian subgroups of $\mathbb{E}_{max}(C, A)$. Since $\mathbf{BiFun}(\mathbb{E}_{max})$ has a maximal element $\mathbb{E}_{max}$, one can show similarly to the proof of 7.16 that the join can also be expressed by

$$F \vee_{bf} F' = \wedge\{G \in \mathbf{BiFun}(\mathbb{E}_{max}) \mid F \leq G,\ F' \leq G\}.$$

$\square$

7.4.1. *Lattice of closed additive sub-bifunctors.* As discussed in Proposition 4.9, for any additive category $\mathcal{A}$ there is a bijection between exact structures and closed additive sub-bifunctors of $\mathbb{E}_{max}$. We already know that the exact structures form a lattice [BHLR, Theorem 5.3]. In this section we define a lattice structure on the class $\mathbf{CBiFun}(\mathcal{A})$ of closed additive sub-bifunctors of $\mathbb{E}_{max}$.

Lemma 7.19. [DRSS, corollary 1.5] Consider a family $\{F_i\}_{i \in I}$ of closed sub-bifunctors of $\mathbb{E}_{max}$. Then the intersection $\cap_{i \in I} F_i$ is a closed sub-bifunctor of $\mathbb{E}_{max}$bifunctor, given by $\{\cap F_i\}(C, A) = \cap\{F_i(C, A)\}$ on objects.

Remark 7.20. If F and F' are closed bifunctors in $\mathbf{CBiFun}(\mathcal{A})$ then their sum $F + F'$ is the sub-bifunctor of $\mathbb{E}_{max}$ given by $\{F + F'\}(C, A) = F(C, A) + F'(C, A)$ on objects. Note that the sum of closed sub-bifunctors is not always closed.

Theorem 7.21. For an additive category $\mathcal{A}$, the closed additive sub-bifunctors of $\mathbb{E}_{max}$ form a complete bounded lattice $(\mathbf{CBiFun}(\mathcal{A}), \leq, \wedge, \vee_{cbf})$.

Proof. The lattice structure is given as follows: the meet is defined by

$$F \wedge F' = F \cap F'$$

while the join is defined by

$$F \vee_{cbf} F' = \cap\{F'' \in \mathbf{CBiFun}(\mathcal{A}) \mid F \leq F'', F' \leq F''\},$$

which is well defined since the intersection is always a non empty, containing $\mathbb{E}_{max}$. Lemma 7.19 ensures that $\mathbf{CBiFun}(\mathcal{A})$ forms a closed meet-semilattice, and the definition of join turns it into a closed lattice, which is bounded by $\mathbb{E}_{min}$ below and $\mathbb{E}_{max}$ above. □

Remark 7.22. The closed sub-bifunctors $(\mathbf{CBiFun}(\mathcal{A}), \leq)$ form a subposet of $(\mathbf{BiFun}(\mathbb{E}_{max}), \leq)$. However, $(\mathbf{CBiFun}(\mathcal{A}, \leq, \wedge, \vee_{cbf})$ is not a sublattice of $\mathbf{BiFun}(\mathbb{E}_{max}), \leq, \wedge, \vee_{bf})$ because their joins are different. In fact, for $F, F' \in \mathbf{CBiFun}(\mathcal{A})$, the join $F \vee_{bf} F' = F + F'$ is not necessarily closed. As discussed in Remark 7.14, the join of $< \alpha >$ with $< \gamma >$ in $\mathbf{BiFun}(\mathbb{E}_{max})$ is $< \alpha, \gamma >$ which is not closed. The join of $< \alpha >$ with $< \gamma >$, in $\mathbf{CBiFun}(\mathcal{A})$ is $< \alpha, \gamma, \delta >$. In general, for $F, F' \in \mathbf{CBiFun}(\mathcal{A})$ we have that $F \vee_{bf} F' \leq F \vee_{cbf} F'$.

7.5. **Lattice of bimodules over the Auslander algebra.** We return now to the study of the bimodule B over the Auslander algebra A defined in Section 4.4. As is the case for any module over a ring, recall that the set $\mathbf{Bim}(\mathrm{B})$ of sub-bimodules of B forms a complete bounded modular lattice

$$(\mathbf{Bim}(\mathrm{B}), \leq, \wedge_{Bim}, \vee_{Bim}),$$

where the meet is given by intersection and the join is given by the sum $N + N'$ of sub-bimodules.

Definition 7.23. An element $N \in \mathbf{Bim}(\mathrm{B})$ is said to be a *closed* bimodule if there exists a *closed* sub-bifunctor F of $\mathrm{Ext}^1_{\mathcal{E}_{max}}$ such that $Ev_X(F) = N$ where

$$Ev_X : \mathbf{CBiFun}(\mathcal{A}) \longrightarrow \mathbf{Bim}(\mathrm{B})$$
$$F \mapsto F(X, X)$$

is the evaluation at the object $X \in \mathcal{A}$.

Lemma 7.24. The intersection of two closed sub-bimodules of B is again closed.

Proof. Let N and P be two closed sub-bimodules of B such that $\Phi(F) = N$ and $\Phi(G) = P$. We consider the sub-bifunctor H of $\mathrm{Ext}^1_{\mathcal{E}_{max}}$ given by the meet of $F \wedge G = H$. By Lemma 7.19, H is closed. Since

$$N \cap P = F(X, X) \cap G(X, X) = H(X, X),$$

the intersection is a closed sub-bimodule of B. □

Theorem 7.25. The subset $\mathbf{Cbim}(\mathrm{B})$ of closed sub-bimodules of B forms a complete bounded lattice

$$(\mathbf{Cbim}(\mathrm{B}), \subseteq, \cap, \vee_{Cbim}).$$

Proof. First this class is a poset ordered by inclusion. Second it is a meet-semi-lattice using the associative, commutative intersection of modules. Third, it is a join-semi-lattice using the following operation

$$\vee_{Cbim} : \mathbf{Cbim}(\mathrm{B}) \times \mathbf{Cbim}(\mathrm{B}) \longrightarrow \mathbf{Cbim}(\mathrm{B})$$

$$(N, P) \mapsto N \vee P = \cap\{R \in \mathbf{Cbim}(B) | N \subset R, P \subset R\}$$

which is associative commutative and satisfies the following property:

$$P \vee (P \wedge N) = N = N \wedge (N \vee P) \text{ for all } N, P \in \mathbf{Cbim}(\mathrm{B}).$$

The intersection in this definition of the join is well defined since the set includes B by assumption. These operations define a lattice structure on $\mathbf{Cbim}(\mathrm{B})$. Since the lattice has a minimal element 0 and a maximal element B, it is a bounded lattice. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ by a family of weakly exact structures in $\mathbf{Cbim}(B)$. Their meet is given by $\bigcap\limits_{\lambda\in\Lambda} N_\lambda$ and the join is given by

$$\cap\{N'' \in \mathbf{Cbim}(\mathrm{B}) \mid N_\lambda \subseteq N'', \ \forall \lambda \in \Lambda\}.$$

Therefore, the lattice is complete. □

In the setting of this subsection, the bimodule $\mathrm{B} = \mathbb{E}_{max}(X, X)$ is finite-dimensional, thus B and all of its submodules have a non-zero socle. We know from Proposition 7.17 that the Auslander-Reiten sequences lie in the socle of the bimodule B, and since all non-projective objects admit an Auslander-Reiten sequence in $\mathcal{A}$ ending there, one can derive that the socle is precisely formed by all Auslander-Reiten sequences in $\mathcal{A}$. Based on Auslander's concept of defects, Enomoto shows in [En18] that the lattice $\mathbf{Cbim}(\mathrm{B})$ is an atomic lattice, in fact it is a boolean lattice determined by its atoms, the Auslander-Reiten sequences in $\mathcal{A}$ (see also [FG20, Theorem 2.26]).

Reformulated in module-theoretic terms, that means that the closed sub-bimodules of $B = \mathbb{E}_{max}(X, X)$ are uniquely determined by their socle, and for every choice of elements in the socle, there is a unique closed sub-bimodule of B having precisely these elements as its socle. If the socle is formed by a set S of Auslander-Reiten sequences, we can thus denote by $\mathbb{E}(S)$ the subbimodule of B determined by S. For all elements $\sigma \in S$, denote by $\mathbb{E}_\sigma$ the bimodule corresponding to the exact structure $\mathcal{E}_\sigma$ introduced in Proposition 7.17. Since the lattice $\mathbf{Cbim}(\mathrm{B})$ is atomic, we conclude that

$$\mathbb{E}(S) = \bigvee_{\sigma\in S} \mathbb{E}_\sigma.$$

There may be several submodules of B with the same socle S, but only one of them is closed. As explained in the proof of [FG20, Theorem 2.26], this closed submodule with socle S corresponds to a Serre subcategory $\mathcal{S}$ generated by the simple objects contained in the set S. All other submodules of B with socle S correspond to certain subcategories of $\mathcal{S}$, but only the closed one is given by the abelian length category formed by all extensions of its simple objects. In other words, $\mathbb{E}(S)$ is maximal, so we derive the following result:

Proposition 7.26. For every set S of Auslander-Reiten sequences, the closed bimodule $\mathbb{E}(S)$ of B introduced above is the maximal submodule of B whose socle is S.

This fact is illustrated nicely in the example in Section 4.3. It is also shown independently for Nakayama algebras in [BHT, Theorem 6.9].

7.6. **Lattice of topologizing subcategories.** Topologizing subcategories of an abelian category $\mathcal{C}$ form a complete lattice. The order is given by the canonical inclusion of categories and the meet is given by the usual intersection. This is a complete semi-lattice and, therefore, it has a canonical join operation upgrading it to a complete lattice. It is straightforward to check from the definitions that the join is given by the closure of the union by finite direct sums:

$$\bigvee : Top(\mathcal{C}) \times Top(\mathcal{C}) \to Top(\mathcal{C})$$
$$(T, T') \longmapsto \oplus\{T \cup T'\}.$$

Since this lattice has a canonical minimal element, it is moreover bounded.

By definition, each Serre subcategory of an abelian category is topologizing. Thus, Serre subcategories form a subposet of the lattice of topologizing subcategories. By similar arguments this subposet admits a lattice structure, with the join given by the closure of the union by finite extensions. Since the closure of the union by finite direct sums is, in general, not extension-closed, the join of Serre subcategories in the lattice of topologizing subcategories is different from their join in the lattice of Serre subcategories. In other words, the lattice of Serre subcategories is a subposet, but not a sublattice of the lattice of all topologizing subcategories.

Given a topologizing subcategory $\mathcal{C}$ of the category $\mathbf{coh}(\mathcal{A})$, its topologizing subcategories in the sense of definition 6.10 form a lattice, which is an interval in the lattice of all topologizing subcategories in $\mathbf{coh}(\mathcal{A})$. Serre subcategories of $\mathcal{C}$ form a lattice, which is an interval in the lattice of all Serre subcategories in $\mathbf{coh}(\mathcal{A})$. It is a subposet, but not a sublattice of the lattice of topologizing subctegories of $\mathcal{C}$.

We formulate this observation explicitly in the case of the categories of defects of weakly extriangulated structures:

Proposition 7.27. Let $\mathcal{A}$ be an essentially small category and $(\mathbb{W}, \mathfrak{s})$ a weakly extriangulated structure on it, then the topologizing subcategories of $\mathbf{def}\,\mathbf{W}$ form a bounded complete lattice

$$(\mathbf{Top}(\mathbf{W}), \subseteq, \bigcap, \bigvee).$$

Serre subcategories of $\mathbf{def}\,\mathbf{W}$ also form a lattice, which is a subposet, but not a sublattice of $(\mathbf{Top}(\mathbf{W}))$.

7.7. **Lattices of extriangulated and weakly extriangulated substructures.** Let $\mathcal{A}$ be an essentially small additive category. We consider the class of all weakly extriangulated structures on $\mathcal{A}$.

Lemma 7.28. Let $\{\mathbb{W}_i\}_{i\in\omega}$ be a family of weakly extriangulated structures on $\mathcal{A}$. Then the intersection $\cap_{i\in\omega}\mathbb{W}_i$ is also a weakly extriangulated structure.

Proof. Similar to Lemma 5.2 of [BHLR]. □

Theorem 7.29. Let $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$ be a weakly extriangulated category. Then all its weakly extriangulated substructures form a bounded complete lattice:

$$(\mathbf{WET}(\mathcal{A}), \leq, \bigwedge, \bigvee)$$

Proof. We consider the set $\mathbf{WET}(\mathcal{A})$ of all the additive sub-bifunctors of $\mathbb{W}$ on the essentially small category $\mathcal{A}$. They are ordered by

$$W \leq W' \iff W(C,A) \subseteq_{Ab} W'(C,A) \text{ for all } A, C \in \mathcal{A}$$

that is, $W(C,A)$ is a subgroup of $W'(C,A)$ for every pair of objects in $\mathcal{A}$. It follows from 7.28 that $(\mathbf{WET}(\mathcal{A}), \leq, \bigwedge)$ is a meet semi-lattice with the meet $(W \bigwedge W')(C,A) = W(C,A) \cap W'(C,A), \forall A, C \in \mathcal{A}$, by using the intersection of abelian groups.

It also forms a join semi-lattice where the join is defined by

$$\mathcal{W} \vee_{\mathbb{W}} \mathcal{W}' = \bigwedge \{\mathcal{V} \in \mathbf{WET}(\mathcal{A}) \mid \mathcal{W} \subseteq \mathcal{V}, \mathcal{W}' \subseteq \mathcal{V}\}$$

This join is well-defined for $\mathbf{WET}(\mathcal{A})$ since the set includes $\mathbb{W}$ by assumption, and so $\mathbf{WET}(\mathcal{A})$ is a complete meet semi-lattice: $\mathbb{W}$ is its unique maximal element. These operations satisfy the axioms of 7.1 and form then a structure of a complete lattice. Moreover the lattice structure defined above on $\mathbf{WET}(\mathcal{A})$ has a minimal element given by the split weakly extriangulated structure $\mathbb{W}_{min}$, so it is a bounded lattice. □

Corollary 7.30. Let $(\mathcal{A}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Then all the additive sub-bifunctors of $\mathbb{E}$ form a bounded complete lattice.

7.8. **Isomorphims of lattices.**

7.8.1. *The three large isomorphic lattices.*

Theorem 7.31. Let $\mathcal{A}$ be an additive category. The map $\Phi : \mathcal{W} \mapsto \mathrm{Ext}^1_{\mathcal{W}}(-,-)$ induces a *lattice isomorphism*

$$(\mathbf{Wex}(\mathcal{E}_{max}), \subseteq, \cap, \vee_{\mathcal{W}}) \;\cong\; (\mathbf{BiFun}(\mathbb{E}_{max}), \leq, \wedge, \vee_{bf}).$$

Proof. We have already shown in Proposition 4.9 that Φ is an isomorphism of posets. We need to verify that it preserves the meet and the join. Let $\mathcal{W}$ and $\mathcal{W}'$ be two weakly exact structures, then $\mathcal{W} \wedge \mathcal{W}'$ is also an exact structure. Let A and C be two objects in $\mathcal{A}$.

$$\begin{aligned}
\mathrm{Ext}^1_{\mathcal{W}\wedge\mathcal{W}'}(C,A) &= \{\, \overline{(i,d)} \mid \; A \xrightarrow{i} B \xrightarrow{d} C \; \in \mathcal{W} \wedge \mathcal{W}'\} \\
&= \{\overline{(i,d)} \mid (i,d) \; \in \mathcal{W}\} \;\cap\; \{\overline{(i,d)} \mid (i,d) \; \in \mathcal{W}'\} \\
&= \mathrm{Ext}^1_{\mathcal{W}}(C,A) \cap \mathrm{Ext}^1_{\mathcal{W}'}(C,A)
\end{aligned}$$

Therefore the two sub-bifunctors $\mathrm{Ext}^1{}_{\mathcal{W}\wedge\mathcal{W}'}(-,-)$ and $\mathrm{Ext}^1_{\mathcal{W}}(-,-)\wedge\mathrm{Ext}^1_{\mathcal{W}'}(-,-)$ coincide, which shows that Φ is a morphism of meet-semilattices. Moreover, the join is defined in both lattices in the same way using intersections (meet), hence Φ is a morphism of lattices. □

Theorem 7.32. Consider the setting of an additively finite category $\mathcal{A}$ as in Section 4.4 and the bimodule B over the Auslander algebra A defined there. Then the evaluation map yields an isomorphism of lattices

$$Ev_X : \mathbf{BiFun}(\mathbb{E}_{max}) \longrightarrow \mathbf{Bim}(\mathrm{B})$$
$$F \mapsto F(X,X)$$

Proof. It is easy to show that the map Ev_X is well defined, injective, surjective, morphism of posets, morphism of lattices and so it induces an isomorphism of lattices. □

Corollary 7.33. If $\mathcal{A}$ is an additively finite, Hom-finite Krull-Schmidt category then the three lattice structures we defined on $\mathbf{Wex}(\mathcal{A})$, $\mathbf{BiFun}(\mathcal{A})$ and $\mathbf{Bim}(B)$ are isomorphic.

Proof. Combine 7.31 and 7.32. □

7.8.2. *The three small isomorphic lattices.*

Theorem 7.34. Let $\mathcal{A}$ be an additive category. The map $\Phi : \mathcal{E} \mapsto \mathrm{Ext}^1_{\mathcal{E}}(-,-)$ induces a *lattice isomorphism* between $(\mathbf{Ex}(\mathcal{A}), \subseteq, \cap, \vee)$ and $(\mathbf{CBiFun}(\mathcal{A}), \leq, \wedge, \vee)$.

Proof. Same as for Theorem 7.31. □

Theorem 7.35. If $\mathcal{A}$ is an additively finite, Hom-finite Krull-Schmidt category then the two lattices $(\mathbf{CBiFun}(\mathcal{A}), \leq, \wedge, \vee_{Cbf})$ and $(\mathbf{Cbim}(B), \subseteq, \cap, \vee_{Cbim})$ are isomorphic.

Proof. As already verified in Theorem 7.32, the evaluation map Ev_X preserves the order and the meet-semi-lattice structure. But the join for closed sub-bimodules is given by intersections on both sides, therefore Ev_X also preserves the join-semi-lattice structure. □

Corollary 7.36. If $\mathcal{A}$ is an additively finite, Hom-finite Krull-Schmidt category then the three lattice structures defined above on $\mathbf{Ex}(\mathcal{A})$, $\mathbf{CBiFun}(\mathcal{A})$ and $\mathbf{Cbim}(\mathrm{B})$ are isomorphic.

Proof. By 7.34 and 7.35. □

7.8.3. *General isomorphism of lattices.*

Proposition 7.37. Let $(\mathcal{A}, \mathbb{W}, \mathfrak{s})$ be a weakly extriangulated category. Then there is a lattice isomophism between the lattice of additive sub-bifunctors of $\mathbb{W}$ and the lattice of topologizing subcategories of $\mathbf{def}\,\mathbb{W}$.

Proof. The proof of [En20, Theorem B], with Step 3 removed, applies word for word. □

Corollary 7.38. Let $\mathcal{W}$ be a weakly exact structure on $\mathcal{A}$. Then there is a lattice isomorphism between the interval $[\mathcal{W}^{\text{add}}, \mathcal{W}]$ in the lattice of weakly exact structures on $\mathcal{A}$ and the lattice of topologizing subcategories of $\mathbf{def}\,\mathbb{W}$.

Corollary 7.39. When the category $\mathcal{A}$ admits a unique maximal weakly exact structure $\mathcal{W}^{\max}$, the lattice of weakly exact structures on $\mathcal{A}$ is isomorphic to the lattice of topologizing subcategories of $\mathbf{def}\,\mathbb{W}^{\max}$.

In particular we get the following summarising result:

Corollary 7.40. Let $\mathcal{A}$ be an idempotent complete essentially small additive category, then the following four lattices are isomorphic:

$$\mathbf{Wex}(\mathcal{A}) \xrightarrow{\sim} \mathbf{BiFun}(\mathcal{A}) \xrightarrow{\sim} \mathbf{Bim}(B) \xrightarrow{\sim} \mathbf{def}\,\mathcal{E}^{\max}.$$

Proof. It follows from 3.17, 7.33 and 7.39. □

Note that when $\mathcal{A}$ is idempotent complete, we can use arguments from [En18, En19, FG20] instead. In particular, this approach would give another proof of the existence of $\mathcal{W}^{\max}$ in this generality.

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